MINOR ARCS FOR GOLDBACH'S PROBLEM

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Abstract. The ternary Goldbach conjecture states that every odd number $n \geq 7$ is the sum of three primes. The estimation of sums of the form $\sum_{p \leq x} e(\alpha p)$, $\alpha = a/q + O(1/q^2)$, has been a central part of the main approach to the conjecture since (Vinogradov, 1937). Previous work required q or x to be too large to make a proof of the conjecture for all n feasible.

The present paper gives new bounds on minor arcs and the tails of major arcs. For $q \geq 4 \cdot 10^6$, these bounds are of the strength needed to solve the ternary Goldbach conjecture. Only the range $q \in [10^5, 4 \cdot 10^6]$ remains to be checked, possibly by brute force, before the conjecture is proven for all n.

The new bounds are due to several qualitative improvements. In particular, this paper presents a general method for reducing the cost of Vaughan's identity, as well as a way to exploit the tails of minor arcs in the context of the large sieve.

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1. Introduction

The ternary Goldbach conjecture (or *three-prime conjecture*) states that every odd number ≥ 7 is the sum of three primes. I. M. Vinogradov [Vin37] showed in 1937 that every odd integer larger than a very large constant C is indeed the sum of three primes. His work was based on the study of exponential sums

$$\sum_{n \leq N} \Lambda(n) e(\alpha n)$$

and their use within the circle method.

Unfortunately, further work has so far reduced C only to e^{3100} ([LW02]; see also [CW89]), which is still much too large for all odd integers up to C to be checked numerically. The main problem has been that existing bounds for (1.1) in the *minor arc* regime – namely, $\alpha = a/q + O(1/q^2)$, $\gcd(a,q) = 1$, q relatively large – have not been strong enough.

The present paper gives new bounds on smoothed exponential sums

$$S_\eta(x,\alpha) = \sum_n \Lambda(n) e(\alpha n) \eta(n/x). \tag{1.1}$$

These bounds are clearly stronger than those on smoothed or unsmoothed exponential sums in the previous literature, including the bounds of [Tao]. (See also work by Ramaré [Ram10].)

In particular, on all arcs around a/q, $q > 4 \cdot 10^6$, the bounds are of the strength required for a full solution to the three-prime conjecture. The same holds on the tails of arcs around a/q for smaller q.

Taken together with existing computational work [Pla11], this means that all that is needed to complete the proof of the three-prime conjecture is a bound for a finite number of arcs, namely, those around a/q, $10^5 < q \leq 4 \cdot 10^6$. Moreover, these arcs are made rather narrow by the estimates given here on tails of arcs. Giving a bound for this narrow range of q's is a task that could be done either by a finite amount of computation – namely, an extension of the computations in [Pla11] to all $q \leq 4 \cdot 10^6$ – or by a further improvement of bounds on exponential sums.

The quality of the results here is due to several new ideas of general applicability. In particular, §4.1 introduces a way to obtain cancellation from Vaughan's identity. Vaughan's identity is a two-log gambit, in that it introduces two convolutions (each of them at a cost of log) and offers a great deal of flexibility in compensation. One of the ideas in the present paper is that at least one of two logs can be successfully recovered after having been given away in the first stage of the proof. This reduces the cost of the use of this basic identity in this and, presumably, many other problems.

We will also see how to exploit being on the tail of a major arc, whether in the large sieve (Lemma 4.3, Prop. 4.5) or in other contexts.

There are also several technical improvements that make a qualitative difference; see the discussions at the beginning of §3 and §4. Considering smoothed sums – now a common idea – also helps. (In the general context of the circle method, smooth sums go back to Hardy-Littlewood; in recent work on the current problem, they appear in [Tao].)

1.1. **Results.** The main bound we are about to see is essentially proportional to $((\log q)/\sqrt{\phi(q)})\cdot x$. The terms δ_0, δ_1 below are there to improve the bound when we are on the tail of an arc.

Main Theorem. *Let $x \geq x_0$, $x_0 = 3.1\cdot 10^{28}$. Let $S_\eta(\alpha, x)$ be as in (1.1), with η defined in (1.4). Let $\alpha = a/q + \delta/x$, $q \leq Q$, $\gcd(a,q) = 1$, $|\delta/x| \leq 1/qQ$, where $Q = x^{0.7}/8$. If $q \leq 2x^{0.3}$, then*

$$\begin{aligned}|S_\eta(\alpha,x)| \leq &\frac{3.2423 + R_{x,\delta_1 q}\log 2\delta_1 q}{\sqrt{\delta_0\phi(q)}}\cdot x + \frac{1.6943 + 0.1108\log 2\delta_1 q}{\sqrt{\delta_0 q}}\cdot x\\ &+ \frac{\min\left(\frac{11}{10}\log x + \frac{1}{2}\log q + 0.5683,\ \frac{76}{9}\log q + 17.489\right)}{\delta_2^2\phi(q)}\cdot x \\ &+ 0.0045\frac{x}{\delta_1 q^2} + 3.336 x^{0.85},\end{aligned} \tag{1.2}$$

where

(1.3)

$$\delta_0 = \max(1, |\delta|/4), \qquad \delta_1 = \begin{cases} 1 & \text{if } |\delta| \leq 4 \\ |\delta|/2 & \text{if } |\delta| > 4,\end{cases} \qquad \delta_2 = \max\left(1, \frac{\pi}{2\sqrt{3}}\delta\right),$$

$$R_{x,t} = 0.4091 + 0.28151\log\left(1 + \frac{\log 50\sqrt{t}}{\log\frac{x^{0.55}}{25 t^{3/2}}}\right).$$

If $q > 2x^{0.3}$, then

$$|S_\eta(\alpha,x)| \leq 0.2974(\log\log x)(\log x)x^{0.85}.$$

For $x \geq x_0$ and $q \leq 2x^{0.3}$, $R_{x,q} \leq R_{x_0,2x^{0.3}} = 0.96188\ldots$ and $R_{x,\delta_1 q} \leq R_{x_0,4x^{0.3}} = 1.11890\ldots$. If $q \sim 4\cdot 10^6$ and $|\delta| \leq 4$ (likely to be the worst situation in practice, given major-arc bounds) then $R_{x,\delta_1 q} \leq R_{x_0,2\cdot 10^6} = 0.6238\ldots$.

The classical choice for η in (1.1) is $\eta(t) = 1$ for $t \leq 1$, $\eta(t) = 0$ for $t > 1$, which, of course, is not smooth, or even continuous. We use

$$\eta(t) = \eta_0(t) = 4\max(\log 2 - |\log 2t|, 0), \tag{1.4}$$

as in Tao [Tao], in part for purposes of comparison. (This is the multiplicative convolution of the characteristic function of an interval with itself.) Nearly all work should be applicable to any other sufficiently smooth function η of fast decay. It is important that $\widehat{\eta}$ decay at least quadratically.

1.2. **History.** The following notes are here to provide some background; no claim to completeness is made.

Vinogradov's proof [Vin37] was based on his novel estimates for exponential sums over primes. Most work on the problem since then, including essentially all work with explicit constants, has been based on estimates for exponential sums; there are some elegant proofs based on cancellation in other kinds of sums ([HB85], [IK04, §19]), but they have not been made to yield practical estimates.

The earliest explicit result is that of Vinogradov's student Borodzin (1939). Vaughan [Vau77] greatly simplified the proof by introducing what is now called Vaughan's identity.

The current record is that of Liu and Wang [LW02]: the best previous result was that of [CW89]. Other recent work falls into the following categories.

Conditional results. The ternary Goldbach conjecture has been proven under the assumption of the generalized Riemann hypothesis [DEtRZ97].

q_0	$\frac{\lvert S_\eta(a/q,x)\rvert}{x}$ here	$\frac{\lvert S_\eta(a/q,x)\rvert}{x}$, [Tao]
10^5	0.08535	0.48556
10^6	0.03165	0.17515
$2 \cdot 10^6$	0.02350	0.13209
$3 \cdot 10^6$	0.01977	0.11220
$4 \cdot 10^6$	0.01751	0.09999
10^7	0.01194	0.06934

TABLE 1. Upper bounds here and in [Tao] on $x^{-1}|S_\eta(a/q,x)|$ for $q \geq q_0$, $|\delta| \leq 4$, $x = 3.1 \cdot 10^{28}$. The trivial bound is ~ 1.

Ineffective results. An example is the bound given by Buttkewitz [But11]. The issue is usually a reliance on the Siegel-Walfisz theorem. In general, to obtain effective bounds with good constants, it is best to avoid analytic results on L-functions with large conductor. (The present paper implicitly uses known results on the Riemann ζ function, but uses nothing at all about other L-functions.)

Results based on Vaughan's identity. Vaughan's identity [Vau77] greatly simplified matters; most textbook treatments are by now based on it. The minor-arc treatment in [Tao] updates this approach to current technical standards (smoothing), while taking advantage of its flexibility (letting variable ranges depend on q).

Results based on log-free identities. Using Vaughan's identity implies losing a factor of $(\log x)^2$ (or $(\log q)^2$, as in [Tao]) in the first step. It thus makes sense to consider other identities that do not involve such a loss. Most approaches before Vaughan's identity involved larger losses, but already [Vin37, §9] is relatively economical, at least for very large x. The work of Daboussi [Dab96] and Daboussi and Rivat [DR01] explores other identities. (A reading of [DR01] gave part of the initial inspiration for the present work.) Ramaré's work [Ram10] – asymptotically the best to date – is based on the Diamond-Steinig inequality (for k large).

* * *

The author's work on the subject leading to the present paper was at first based on the (log-free) Bombieri-Selberg identity ($k = 3$), but has now been redone with Vaughan's identity in its foundations. This is feasible thanks to the factor of log regained in §4.1.

1.3. **Comparison to earlier work.** Table 1 compares the bounds for the ratio $|S_\eta(a/q,x)|/x$ given by this paper and by [Tao] for $x = 3.1 \cdot 10^{26}$ and $q \geq q_0$, where q_0 is 10^5, 10^6 or $4 \cdot 10^6$. We are comparing worst cases: $\phi(q)$ as small as possible (q divisible by $2 \cdot 3 \cdot 5 \cdots$) in the result here, and q divisible by 4 (implying $4\alpha \sim a/(q/4)$) in Tao's result. The main term in the result in this paper improves slowly with increasing x; the results in [Tao] worsen slowly with increasing x.

We are focusing on a comparison with [Tao] for two reasons: (a) the results in [DR01] are unfortunately worse than the trivial bound in this range, and (b) Ramaré's result [Ram10, Thm. 3] is not applicable in this range, since its condition $\log q \leq (1/50)(\log x)^{1/3}$ is not satisfied.

The qualitative gain with respect to [Tao] is about $\log(q)\sqrt{\phi(q)/q}$, which is $\sim \log(q)/\sqrt{e^\gamma(\log\log q)}$ in the worst case.

As we shall see, Table 1 suggests that the main theorem in the present paper will be enough to prove the ternary Goldbach conjecture once the zeroes of

Dirichlet L-functions of conductor $10^5 < q \leq 4 \cdot 10^6$ are checked up to a moderate height. (Since the bound in the main theorem depends on $1/\sqrt{\phi(q)}$ rather than $1/sqrtq$, it would actually be enough to check all conductors $10^5 < q \leq 10^6$ and some conductors $10^6 < q \leq 4 \cdot 10^6$, e.g., those that are divisible by 30.)

1.4. **Existing numerical work.** The numerical inputs we can count on are as follows

- every odd number $n \leq n_0$ is the sum of three primes, where
$$n_0 = 6.15697 \cdot 10^{28}.$$
This is a moderately cautious choice for n_0, in that there are statements in the literature that would enable one to take n_0 higher; see Appendix A.
- L-functions of conductor $q \leq 10^5$ have no zeroes outside the critical line up to height $|\Im(z)| \leq 10^8/q$ (D. Platt's thesis [Pla11]).

The estimates in this paper for large δ, together with D. Platt's results on zeroes of L-functions of small conductor, should be enough to give major-arc bounds for $q \leq 10^5$.

Running Platt's program to extend his results to all conductors $q \leq Q_0$ up to height $|\Im(z)| \leq Q_0 T/q$ with $T = O(Q_0)$ would take time $O(Q^3 \log^2 Q)$; considering a single q takes time $O(QT \log q \log QT)$ (D. Platt, personal communication). Since the resources needed for $Q_0 = 10^5$ were relatively modest, this would seem to put the task of checking all q up to 10^6 and some q up to $4 \cdot 10^6$ just about on the limit of what is nowadays feasible by a purely computational approach.

1.5. **Major and minor arcs: a summary assessment.** In the circle method, the number of representations of a number N as the sum of three primes is represented as an integral over the "circle" $\mathbb{R}/\mathbb{Z}$, which is partitioned into major arcs $\mathfrak{M}$ and minor arcs $\mathfrak{m}$:

$$\begin{aligned}\sum_{n_1+n_2+n_3=N} \Lambda(n_1)\Lambda(n_2)\Lambda(n_3) &= \int_{\mathbb{R}/\mathbb{Z}} (S(\alpha,x))^3 e(-N\alpha) d\alpha \\ &= \int_{\mathfrak{M}} (S(\alpha,x))^3 e(-N\alpha) d\alpha + \int_{\mathfrak{m}} (S(\alpha,x))^3 e(-N\alpha) d\alpha.\end{aligned} \tag{1.5}$$

The major arcs $\mathfrak{M}$ consist of intervals around the rationals a/q, $q \leq Q_0$. In previous work[1], Q_0 grew with x; in our setup, Q_0 is a constant. Smoothing changes the left side of (1.5) into a weighted sum, but, since we aim at an existence result rather than at an asymptotic for the number of representations $p_1+p_2+p_3$ of N, this is obviously acceptable.

1.5.1. *Minor arcs.* The main theorem bounds $\max_{\alpha\in\mathfrak{m}} |S(x,\alpha)|$. The ternary Goldbach conjecture will be proven once we obtain an upper bound for the minor-arc integral $\left|\int_{\mathfrak{m}} |S(x,\alpha)|^3 d\alpha\right|$ that is smaller than a lower bound for the major-arc integral $\int_{\mathfrak{M}} S(x,\alpha)^3 e(-N\alpha) d\alpha$.

(See §6 for a more detailed discussion.)

Traditionally, the upper bound for the minor-arc integral $\left|\int_{\mathfrak{m}} |S(x,\alpha)|^3 d\alpha\right|$ took the form

$$\left|\int_{\mathfrak{m}} |S(x,\alpha)|^3 d\alpha\right| \leq (\max_{\alpha\in\mathfrak{m}} |S(\alpha,x)|_\infty) \cdot |S|_2^2. \tag{1.6}$$

[1]Ramaré's work [Ram10] is in principle strong enough to allow Q_0 to be an unspecified large constant. Tao's work [Tao] reaches this standard only for x of moderate size.

One can of course improve on this by taking out the major-arc contribution:

$$\left|\int_{\mathfrak{m}} |S(x,\alpha)|^3 d\alpha\right| \leq (\max_{\alpha\in\mathfrak{m}} |S(\alpha,x)|_\infty) \cdot \left(|S|_2^2 - \int_{\mathfrak{M}} |S(x,\alpha)|^2 d\alpha\right). \tag{1.7}$$

More crucially, an idea of Heath-Brown's (based on Montgomery's inequality [Mon68], [Hux72]; see [Tao, Lemma 4.6]) allows us, in essence, to multiply the term $|S|_2^2$ in (1.6) or (1.7) by a factor that, while only slightly less than 1 for $x \sim x_0 = 3.1 \cdot 10^{28}$ and $Q_0 \sim 4 \cdot 10^6$, grows like $O(1/\log x)$ as x grows. It is this that allows us to let Q_0 be a constant.

A brief numerical check shows that, together,

- the factor coming from Heath-Brown,
- the fact that $R_{x,\delta_1 q}$ in (1.2) decreases as x increases,

are enough to make the situation for $x \geq x_0$, $q \sim 4 \cdot 10^6$ better when $x \ggg x_0$ than when $x \sim x_0$. Thus we can focus on $x \sim x_0$ as the worst case.

In summary, because $x \sim x_0$ is the worst case and because we can reduce a little the factor of $|S|_2^2 \sim \log x_0$ in (1.6), a bound of slightly less than $0.02x$ on the minor arcs is necessary and sufficient for the minor-arc contribution to be acceptable. By Table 1, this corresponds to $Q_0 = 3 \cdot 10^6$; we choose $Q_0 = 4 \cdot 10^6$ to allow for losses in the major-arc part of the argument.

1.5.2. *Major arcs.* Since Q_0 is a constant, the treatment of the major arcs is a finite problem that can be attacked computationally. The actual estimation of major arcs is a subject for a separate paper; we will simply discuss very briefly what the actual size of the term being estimated should be.

We can choose a different smoothing η_1 for the sum whose ℓ_2 norm is taken (in (1.6) or (1.7)), as opposed to the smoothing η_0 in the sum whose ℓ_∞ norm we bound in the main theorem.

For at least moderately large Q_0 (as is the case here), the major-arc contribution is close to what is believed to be the actual weighted number of representations of N:

$$\begin{aligned}\int_{\mathfrak{M}} (S_{\eta_0}(\alpha,x))(S_{\eta_1}(\alpha,x))^2 d\alpha &\sim \sum_{n_1,n_2,n_3} \Lambda(n_1)\Lambda(n_2)\Lambda(n_3)\eta_0(n/x)\eta_1(n/x)\eta_1(n/x)\\ &\sim 2\prod_{p\geq 3}\left(1 - \frac{1}{(p-1)^2}\right) \cdot (\eta_0 * \eta_1 * \eta_1)(\beta)\\ &= 1.3203\ldots \cdot (\eta_0 * \eta_1 * \eta_1)(\beta),\end{aligned} \tag{1.8}$$

where $\beta = N/x$.

Recall that the minor-arc estimate is proportional to $|S_{\eta_1}|_2^2$, and hence to $|\eta_1|_2^2$. Hence, we should choose β (within a given range) so as to make

$$\frac{\eta_0 * \eta_1 * \eta_1(\beta)}{|\eta_1|_2^2} \tag{1.9}$$

as large as possible; this should also play a role in our choice of η_1. It is easy to see that (1.9) is at most $|\eta_1 * \eta_1|_\infty \cdot |\eta_0|_1/|\eta_1|_2^2 \leq |\eta_0|_1 = 1$; the optimal value of 1 is approached when η_1 is roughly symmetric (additively, around some positive value) and the support of η_1 is somewhat broader than that of η_0.

There are other matters in play: it is also desirable to choose an η_1 whose Mellin transform decays very rapidly, as this minimizes the error term in the

major-arcs estimate. This means we should be content with a good enough η_1, rather than search for an η_1 that is optimal according to a single criterion.

1.5.3. *Conclusion.* The minor-arc estimate, without the substraction of a major-arc term as in (1.7), is

$$1.163699\ldots \ldots \cdot \frac{|S_{\eta_1}|_2^2}{\log x} \sim 1.163\ldots \cdot |\eta_1|_2^2 \quad (1.10)$$

for $Q_0 = 4 \cdot 10^6$ and $x \geq x_0 = 3.1 \cdot 10^{28}$. Substracting a major-arc term would reduce this.

The major-arc integral is close to

$$1.3203\ldots \cdot (\eta_0 * \eta_1 * \eta_1)(\beta). \quad (1.11)$$

If the total losses from an error term here and the ratio (1.9) are less than 10 percent, the three-prime theorem will be proven. Larger errors can obviously be allowed if Q_0 is set higher.

1.5.4. *Final remarks.* Larger losses will be allowable if q is set larger than $4 \cdot 10^6$ or the effect of the substraction of a major-arc term from (1.10) is substantial. It is likely possible to omit all odd values of q between $2 \cdot 10^6$ and $4 \cdot 10^6$ from consideration, as the terms $\phi(q)$ in (1.2) are then closer to q. A technical idea in [Tao] (using 4α instead of α, and summing over odd numbers in some contexts) is also likely to help for q odd, or $q \equiv 2 \bmod 4$.

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2. Preliminaries

2.1. **Notation.** Given positive integers m, n, we say $m|n^\infty$ if every prime dividing m also divides n. We say a positive integer n is *square-full* if, for every prime p dividing n, the square p^2 also divides n. (In particular, 1 is square-full.) We say n is *square-free* if $p^2 \nmid n$ for every prime p. For p prime, n a non-zero integer, we define $v_p(n)$ to be the largest non-negative integer α such that $p^\alpha|n$.

When we write $\sum_n$, we mean $\sum_{n=1}^\infty$, unless the contrary is stated. As usual, μ, Λ, τ and σ denote the Moebius function, the von Mangoldt function, the divisor function and the sum-of-divisors function, respectively.

As is customary, we write $e(x)$ for $e^{2\pi i x}$. The Fourier transform on $\mathbb{R}$ is normalized here as follows:

$$\widehat{f}(t) = \int_{-\infty}^{\infty} e(-xt) f(x) dx.$$

We write $|f|_r$ for the L_r norm of a function f.

We write $O_\leq(R)$ to mean a quantity at most R in absolute value. We quote the following standard bound from [Tao, Lemma 3.1]: for $\alpha \in \mathbb{R}/\mathbb{Z}$ and $f : \mathbb{R} \to \mathbb{C}$ compactly supported, continuous and piecewise C^1,

$$\left|\sum_{n\in\mathbb{Z}} f(n)e(\alpha n)\right| \leq \min\left(|f|_1 + \frac{1}{2}|f'|_1, \frac{\frac{1}{2}|f'|_1}{|\sin(\pi\alpha)|}\right). \tag{2.1}$$

When we need to estimate $\sum_n f(n)$ precisely, we will use the Poisson summation formula:

$$\sum_n f(n) = \sum_n \widehat{f}(n).$$

We will not have to worry about convergence here, since we will apply the Poisson summation formula only to compactly supported functions f whose Fourier transforms decay at least quadratically.

If f is compactly supported and C^2, then the Fourier transform of f does decay at least quadratically:

$$\begin{aligned}\widehat{f}(t) &= \int_{-\infty}^{\infty} f(x)e(-tx)dx = -\int_{-\infty}^{\infty} f'(x)\frac{e(-tx)}{-2\pi i t}dx\\ &= \int_{-\infty}^{\infty} f''(x)\frac{e(-tx)}{(2\pi i t)^2}dx = O_\leq\left(\frac{1}{(2\pi t)^2}|f''|_1\right).\end{aligned} \tag{2.2}$$

This can still be made sense of when f'' is undefined at a finite number of points, provided f is understood as a distribution (and f' has finite total variation). Of course, $|\widehat{f}(t)| \leq |f|_1$, as always.

For the smoothing function η_0 in (1.4),

$$|\eta_0|_1 = 1, \quad |\eta_0'|_1 = 8\log 2, \quad |\eta_0''|_1 = 48, \tag{2.3}$$

as per [Tao, (5.9)–(5.13)]. Similarly, for $\eta_{1,\rho}(t) = \log(\rho t)\eta_0(t)$, where $\rho \geq 4$,

$$\begin{aligned}|\eta_{1,\rho}|_1 &< \log(\rho)|\eta_0|_1 = \log(\rho)\\ |\eta_{1,\rho}'|_1 &= 2\eta_{1,\rho}(1/2) = 2\log(\rho/2)\eta_0(1/2) < (8\log 2)\log\rho,\\ |\eta_{1,\rho}''|_1 &= 4\log(\rho/4) + |2\log\rho - 4\log(\rho/4)| + |4\log 2 - 4\log\rho|\\ &\quad + |\log\rho - 4\log 2| + |\log\rho| < 48\log\rho.\end{aligned} \tag{2.4}$$

Write $\log^+ x$ for $\max(\log x, 0)$.

2.2. **Bounds on sums of $\mu(m)$ and $\Lambda(n)$.** We will need explicit bounds on $\sum_{n\leq N}\mu(n)/n$ and related sums involving μ. The situation here is less well-developed than for sums involving Λ. The main reason is that the complex-analytic approach to estimating $\sum_{n\leq N}\mu(n)$ would involve $1/\zeta(s)$ rather than $\zeta'(s)/\zeta(s)$, and thus strong explicit bounds on the residues of $1/\zeta(s)$ would be needed.

Fortunately all we need is a saving of $(\log n)$ or $(\log n)^2$ on the trivial bound. This is provided by the following.

(1) (Granville-Ramaré [GR96], Lemma 10.2)

$$\left|\sum_{n\leq x:\gcd(n,q)=1}\frac{\mu(n)}{n}\right| \leq 1 \tag{2.5}$$

for all x, $q \geq 1$,

(2) (Ramaré [Ramc]; cf. El Marraki [EM95], [EM96])

$$\left|\sum_{n\leq x}\frac{\mu(n)}{n}\right| \leq \frac{0.03}{\log x} \tag{2.6}$$

for $x \geq 11815$.

(3) (Ramaré [Ramb])

$$\sum_{n\leq x:\gcd(n,q)=1}\frac{\mu(n)}{n} = O_{\leq}\left(\min\left(2.02\frac{q}{\phi(q)}\frac{1}{\log x/q},1\right)\right) \tag{2.7}$$

for all x and all $q \leq x$;

$$\sum_{n\leq x:\gcd(n,q)=1}\frac{\mu(n)}{n}\log\frac{x}{n} = O_{\leq}\left(1.4+\frac{q}{\phi(q)}\right) \tag{2.8}$$

for all x and all q.

Improvements on these bounds would lead to improvements on type I estimates, but not in what are the worst terms overall at this point.

By numerical work carried out by Olivier Ramaré and reproduced independently by the author,

$$\left|\sum_{n\leq x}\frac{\mu(n)}{n}\right| \leq \sqrt{\frac{2}{x}} \tag{2.9}$$

for all real $x \geq 10^{10}$. (In fact, the stronger bound $\leq 1/2\sqrt{x}$ holds for all $3 \leq x \leq 7727068587$, but not for $x = 7727068588 - \epsilon$. The author's verification of (2.9) goes up to 10^{11}.)

We will make reference to various bounds on $\Lambda(n)$ in the literature. The following recent bound ([Rama], supplemented by a numerical check for very small x) will be useful:

$$\sum_{n\leq x}\frac{\Lambda(n)}{n} \leq \log x. \tag{2.10}$$

We also use the following older bounds:

(1) By the second table in [RR96, p. 423], supplemented by a computation for $2\cdot 10^6 \leq V \leq 4\cdot 10^6$,

$$\sum_{n\leq y}\Lambda(n) \leq 1.0004y \tag{2.11}$$

for $y \geq 2\cdot 10^6$.

(2)

$$\sum_{n\leq y}\Lambda(n) < 1.03883y \tag{2.12}$$

for every $y > 0$ [RS62, Thm. 12].

For all $y > 663$,

$$\sum_{n\leq y}\Lambda(n)n < 1.03884\frac{y^2}{2}, \tag{2.13}$$

where we use (2.12) and partial summation for $y > 200000$, and a computation for $663 < y \leq 200000$. It is also true that

$$\sum_{y/2<p\leq y} (\log p)^2 \leq \frac{1}{2} y(\log y) \tag{2.14}$$

for $y \geq 117$: this holds for $y \geq 2 \cdot 758699$ by [RS75, Cor. 2] (applied to $x = y$, $x = y/2$ and $x = 2y/3$) and for $117 \leq y < 2 \cdot 758699$ by direct computation.

2.3. **Basic setup.** We begin by applying Vaughan's identity [Vau77]: for any function $\eta : \mathbb{R} \to \mathbb{R}$, any completely multiplicative function $f : \mathbb{Z}^+ \to \mathbb{C}$ and any $x > 0$, $U, V \geq 0$,

$$\sum_n \Lambda(n) f(n) e(\alpha n) \eta(n/x) = S_{I,1} - S_{I,2} + S_{II} + S_{0,\infty}, \tag{2.15}$$

where

$$\begin{aligned} S_{I,1} &= \sum_{m\leq U} \mu(m) f(m) \sum_n (\log n) e(\alpha m n) f(n) \eta(mn/x), \\ S_{I,2} &= \sum_{d\leq V} \Lambda(d) f(d) \sum_{m\leq U} \mu(m) f(m) \sum_n e(\alpha d m n) f(n) \eta(dmn/x), \\ S_{II} &= \sum_{m>U} f(m) \left(\sum_{\substack{d>U \\ d|m}} \mu(d) \right) \sum_{n>V} \Lambda(n) e(\alpha m n) f(n) \eta(mn/x), \\ S_{0,\infty} &= \sum_{n\leq V} \Lambda(n) e(\alpha n) f(n) \eta(n/x). \end{aligned} \tag{2.16}$$

The proof is essentially an application of the Möbius inversion formula; see, e.g., [IK04, §13.4]. In practice, we will use the function

$$f(n) = \begin{cases} 1 & \text{if } \gcd(n, w) = 1, \\ 0 & \text{otherwise,} \end{cases}$$

where w is a small positive, square-free integer. Then

$$S_\eta(x, \alpha) = S_{I,1} - S_{I,2} + S_{II} + S_{0,\infty} + S_{0,w},$$

where $S_\eta(x, \alpha)$ is as in (1.1) and

$$S_{0,w} = \sum_{n|w} \Lambda(n) e(\alpha n) \eta(n/x).$$

The sums $S_{I,1}$, $S_{I,2}$ are called "of type I" (or linear), the sum S_{II} is called "of type II" (or bilinear). The sum S_0 is in general negligible; for our later choice of V and η, it will be in fact 0. The sum $S_{0,w}$ will be negligible as well.

3. Type I

There are here three main improvements in comparison to standard treatments:

(1) The terms with m divisible by q get taken out and treated separately by analytic means. This all but eliminates what would otherwise be the main term.
(2) For large m, the other terms get handled so as to replace a factor of $\log q$ by a factor of $\log \min(W/U, q)$.

(3) The "error" term $\delta/x = \alpha - a/q$ is used to our advantage. This happens both through the Poisson summation formula and through the use of two successive approximations.

3.1. **Trigonometric sums.** The following is a Vinogradov-type lemma (occupying roughly the middle ground between Lemma 8a and Lemma 8b in [Vin04, Ch. I]).

Lemma 3.1. *Let* $\alpha = a/q + \beta/qQ$, $\gcd(a,q) = 1$, $|\beta| \leq 1$, $Q \geq q$. *Then, for* $U \geq 1$,

$$\sum_{y<n\leq y+q} \min\left(U, \frac{1}{|\sin(\pi\alpha n)|}\right) \leq \min\left(2U + \frac{2q}{\pi}\log 2q, 3U + \frac{2q}{\pi}\log 2eU\right). \tag{3.1}$$

In contrast to the standard procedure, we will sometimes choose to bound some terms by U and other terms by $1/|\sin(\pi\alpha n)|$. This simple change will allow us – in essence – to replace a factor of $\log q$ by a factor of $\log U$ whenever this is to our advantage (see (3.1)). We otherwise follow Daboussi and Rivat [DR01, Lemma 1], with minor improvements.

Proof. We write $m_0 = [y]+[(q+1)/2]$, $j = n-m_0$, so that j ranges in the interval $(-q/2, q/2]$. Then

$$\alpha n = \alpha m_0 + \frac{aj}{q} + \frac{\beta j}{qQ} \equiv \frac{aj}{q} + \delta_0 + \delta_1(j) \bmod 1,$$

where $\delta_0 = \alpha m_0$ is fixed and $\delta_1(j) = \beta j/qQ$ satisfies $|\delta_1(j)| \leq 1/2Q \leq 1/2q$. Write δ_0 in the form $c/q + \delta_2$, where c is an integer and $|\delta_2| \leq 1/2q$. As j runs over the interval $-q/2 < j \leq q/2$, the variable $r = aj + c \bmod q$ occupies each residue class mod p exactly once. We may assume $\delta_2 \geq 0$ without loss of generality (since we can replace α by $-\alpha$ if needed). We bound the terms corresponding to $r = 0$ and $r = -1$ by U each; we will now consider the other terms.

One option (advantageous for U large, but always valid) is to bound every term $\min(U, 1/|\sin(\phi\alpha n)|)$ by $1/|\sin(\phi\alpha n)|$ from now on. The terms corresponding to $r = -k$ and $r = k-1$ ($2 \leq k \leq q/2$) contribute at most

$$\frac{1}{\sin\frac{\pi}{q}(-k+1/2+q\delta_2)} + \frac{1}{\sin\frac{\pi}{q}(k-3/2+q\delta_2)} \leq \frac{1}{\sin\frac{\pi}{q}(k-1/2)} + \frac{1}{\sin\frac{\pi}{q}(k-3/2)},$$

since $x \mapsto 1/\sin x$ is an even function that is convex-up on $(0,\infty)$. Hence the terms with $r \neq 0, -1$ contribute in total

$$\frac{1}{\sin(\pi/2q)} + 2\sum_{2\leq r\leq q/2} \frac{1}{\sin\frac{\pi}{q}(r-1/2)} \leq \frac{1}{\sin(\pi/2q)} + 2\int_1^{q/2} \frac{1}{\sin\frac{\pi}{q}x} dx,$$

where we use again the convexity of $x \mapsto 1/\sin x$. Now

$$\int_1^{q/2} \frac{1}{\sin\frac{\pi}{q}x} dx \leq \frac{q}{\pi}\log\cot\frac{\pi}{2q} \leq \frac{q}{\pi}\log\frac{2q}{\pi},$$

and since we may assume $q \geq 3$ (the statement of the lemma is trivial otherwise) we have $1/\sin(\pi/2q) + (2q/\pi)\log(2q/\pi) \leq (2q/\pi)\log 2q$.

The alternative is to use the bound $\leq U$ for some terms with $r \neq 0, -1$. We consider this only when $U \leq 1/\sin(\pi/2q)$. Assume $U \geq 1$ (or else the estimate

$\leq Uq$ holds and is optimal for trivial reasons). The terms with $r \neq 0, -1$ then sum up to at most

$$\begin{aligned} &U + \sum_{2 \leq r' \leq q/2} \min\left(U, \frac{1}{\sin \frac{\pi}{q}(r' - 1/2 - q\delta_2)}\right) \\ &+ \sum_{2 \leq r' \leq q/2} \min\left(U, \frac{1}{\sin \frac{\pi}{q}(r' - 1/2 + q\delta_2)}\right). \end{aligned} \tag{3.2}$$

We choose to bound a term $1/\sin((\pi/q)(r' - 1/2 \pm q\delta_2))$ by U if and only if $1/\sin((\pi/q)(r' - 1 \pm q\delta_2)) \geq U$. The number of such terms is $\leq (q/\pi)\arcsin(1/U) \mp q\delta_2$, i.e., $(2q/\pi)\arcsin(1/U)$ in total. (Note we are using $(q/\pi)\arcsin(1/U) \geq 1/2$.) Each other term gets bounded by the integral of $1/\sin(\phi\alpha/q)$ from $r' - 1 \pm q\delta_2$ $(> (q/\pi)\arcsin(1/U))$ to $r' \pm q\delta_2$, by convexity. Thus (3.2) is at most

$$\begin{aligned} U + \frac{2q}{\pi} U \arcsin\frac{1}{U} + 2\int_{\frac{q}{\pi}\arcsin(1/U)}^{q/2} \frac{1}{\sin\frac{\pi}{t}} dt \\ \leq U + \frac{2q}{\pi} U \arcsin\frac{1}{U} + 2\frac{q}{\pi} \log\cot\left(\frac{1}{2}\arcsin\frac{1}{U}\right) \\ \leq U + \frac{2q}{\pi}(1 + \log(2U)). \end{aligned}$$

A quick comparison shows that $U + (2q/\pi)(1 + \log(2U)) \leq (2q/\pi)\log 2q$ when $U < 0.2488\ldots$ times q. Note $U < q/4$ implies $U \leq 1/\sin(\pi/2q)$, so the bound $U + (2q/\pi)(1 + \log(2U))$ is certainly valid for $U < q/4$. □

Lemma 3.2. *Let $\alpha = a/q + \beta/qQ$, $\gcd(a, q) = 1$. $|\beta| \leq 1$, $Q \geq q$. Let $y_2 > y_1 \geq 0$. If $y_2 - y_1 \leq q$ and $y_2 \leq Q/2$, then, for any $U \geq 1$,*

$$\begin{aligned} \sum_{\substack{y_1 < n \leq y_2 \\ q \nmid n}} \min\left(U, \frac{1}{|\sin(\pi\alpha n)|}\right) &\leq \min\left(\frac{2q}{\pi}\log 2eq, 2U + \frac{2q}{\pi}\log 2eU\right) \\ &< \frac{2q}{\pi}\log 16U. \end{aligned} \tag{3.3}$$

This is a variant of Lemma 3.2 with the term for n divisible by q set aside. The bound $(2q/\pi)\log 2eq$ is smaller than $2U + (2q/\pi)\log 2eU$ for U of size roughly $q/3$ or larger.

Proof. For $n \leq y_2$, αn equals $an/q + \beta n/qQ = an/q + O_\leq(1/2q)$. We proceed as in the proof of Lemma 3.1, only without excluding the terms labelled $r = 0, -1$ there; the sum we obtain is at most

$$2\sum_{r=1}^{q/2} \min\left(U, \frac{1}{\sin\frac{\pi}{q}(r - 1/2)}\right). \tag{3.4}$$

The estimate without U's is $2/\sin(\pi/2q) + (2q/\pi)\log(2q/\pi) \leq (2q/\pi)\log 2eq$. For the estimate with U's, proceed as before, using the convexity of $1/\sin$ and

choosing U whenever $r \leq 1 + (q/\pi)\arcsin(1/U)$: (3.4) is at most

$$\begin{aligned}2U\left(1+\frac{q}{\pi}\arcsin(1/U)\right) &+ 2\int_{(q/\pi)\arcsin(1/U)}^{q/2} \frac{1}{\sin\frac{\pi}{q}t}dt\\ &\leq 2U + \frac{2q}{\pi}\left(U\arcsin\frac{1}{U} + \log\cot\frac{1}{2}\arcsin\frac{1}{U}\right)\\ &\leq 2U + \frac{2q}{\pi}\log 2eU.\end{aligned}$$

To obtain the last line of (3.3), note that, if $q \geq (16/2e)U$, then $2U \leq \frac{4e}{16} < (2/\pi)(\log 16 - \log 2e)$. □

3.2. **Type I estimates.** Our main type I estimate is the following. One of the main innovations is the manner in which the "main term" (m divisible by q) is separated; we are able to keep error terms small thanks to the particular way in which we switch between two different approximations.

Lemma 3.3. *Let $\alpha = a/q + \delta/x$, $\gcd(a,q) = 1$, $|\delta/x| \leq 1/q^2$, $q \leq x$. Let η be continuous, piecewise C^2 and compactly supported, with $\eta'' \in L_1$. Let $w \in \mathbb{Z}^+$ be square-free. Then, for $D \leq x/w$ and any $\epsilon \in (0,1]$ such that $\epsilon/(1+\epsilon) > 2|\delta q|/x$,*

$$\begin{aligned}\sum_{\substack{m\leq D\\ \gcd(m,w)=1}} \mu(m) \sum_{\substack{n\\ \gcd(n,w)=1}} e(\alpha m n)\eta(mn/x)\qquad&\\ = \frac{x\mu(q)}{q}\frac{\phi(w/q^+)}{w/q^+}\widehat{\eta}(-\delta)\cdot &\sum_{\substack{m\leq M/q^-\\ \gcd(m,wq^-)=1}}\frac{\mu(m)}{m}\\ + O_{\leq}\Bigg(&\frac{D(D+q)\sigma(w)}{xq/w}\cdot\left(\frac{1}{8}-\frac{1}{2\pi^2}\right)|\eta''|_1\\ + |\eta_1'|\cdot&\left(\frac{q}{\pi}\log 2eq + \frac{Dw}{\pi}\left(2(1+\epsilon)\log\frac{2ecx}{D} + \log\frac{16ecx}{Dw}\right)\right)\\ + (2|\eta_1| + |\eta_1'|)\cdot&\left(\frac{3wcx}{Q} + \frac{3(1+\epsilon)}{2\epsilon}\frac{x}{Q}\log\frac{Dw}{M}\right)\Bigg),\end{aligned}\tag{3.5}$$

where $c = 2|\eta|_1/|\eta'|_1 + 1$, $Q = \lfloor x/|\delta q|\rfloor$, $M = \min(\lfloor D\rfloor, Q/2w)$, $q^+ = \gcd(q,w)$ and $q^- = q/q^+$.

A few words on error terms and "recommended usage". The last two lines in (3.5) are usually very small. The term $D\log((4ecx)/D)$ is independent of our choice of approximation; it forces us to have D smaller than x times a constant if we want an error smaller than x. If a, q are chosen so that $|\alpha - a/q| \leq 1/qQ_0$ for some Q_0, then $Q = \lfloor x/|\delta q|\rfloor \geq Q_0$. If $\delta = 0$, we set $Q = \infty$.

Proof. Since $Q = \lfloor x/|\delta q|\rfloor$, we have $\alpha = a/q + O_{\leq}(1/qQ)$ and $q \leq Q$. (If $\delta = 0$, we omit what follows.) Let $Q' = \lceil(1+\epsilon)Q\rceil \geq Q+1$. Then α is *not* $a/q + O_{\leq}(1/qQ')$, and so there must be a different approximation a'/q', $\gcd(a',q') = 1$, $q' \leq Q'$ such that $\alpha = a'/q' + O_{\leq}(1/q'Q')$ (since such an approximation always exists). Obviously, $|a/q - a'/q'| \geq 1/qq'$, yet, at the same time, $|a/q - a'/q'| \leq 1/qQ + 1/q'Q' \leq 1/qQ + 1/((1+\epsilon)q'Q)$. Hence $q'/Q + q/((1+\epsilon)Q) \geq 1$, and so $q' \geq Q - q/(1+\epsilon) \geq (\epsilon/(1+\epsilon))Q$. (Note also that $(\epsilon/(1+\epsilon))Q \geq (2|\delta q|/x)\cdot\lfloor x/\delta q\rfloor > 1$, and so $q' \geq 2$.)

By Möbius inversion,

$$\sum_{\substack{m\leq D\\ \gcd(m,w)=1}} \mu(m) \sum_{\substack{n\\ \gcd(m,w)=1}} e(\alpha mn)\eta(mn/x) = \sum_{\substack{m\leq D\\ \gcd(m,w)=1}} \mu(m) \sum_{d|w} \mu(d) \sum_n e(\alpha mdn)\eta(mdn/x) \tag{3.6}$$

Let $M = \min(\lfloor D\rfloor, Q/2w)$. We start by separating the terms with $m \leq M$, md divisible by q. Then $e(\alpha mdn)$ equals $e((\delta md/x)n)$. By Poisson summation,

$$\sum_n e(\alpha mdn)\eta(mdn/x) = \sum_n \widehat{f}(n),$$

where $f(u) = e((\delta md/x)u)\eta((md/x)u)$. Now

$$\widehat{f}(n) = \int e(-un)f(u)du = \frac{x}{md}\int e\left(\left(\delta - \frac{xn}{md}\right)u\right)\eta(u)du = \frac{x}{md}\widehat{\eta}\left(\frac{x}{md}n - \delta\right).$$

By assumption, $md \leq Mw \leq Q/2 \leq x/2|\delta q|$, and so $|x/md| \geq 2|\delta q| \geq 2\delta$. Thus, by (2.2),

$$\begin{aligned}\sum_n \widehat{f}(n) &= \frac{x}{md}\left(\widehat{\eta}(-\delta) + \sum_{n\neq 0}\widehat{\eta}\left(\frac{nx}{md} - \delta\right)\right)\\ &= \frac{x}{md}\left(\widehat{\eta}(-\delta) + O_\leq\left(\sum_{n\neq 0}\frac{1}{\left(2\pi\left(\frac{nx}{md}-\delta\right)\right)^2}\right)\cdot |\eta''|_1\right)\\ &= \frac{x}{md}\widehat{\eta}(-\delta) + \frac{md}{x}\frac{|\eta''|_1}{(2\pi)^2}O_\leq\left(\max_{|r|\leq 1/2}\sum_{n\neq 0}\frac{1}{(n-r)^2}\right).\end{aligned} \tag{3.7}$$

Since $x \mapsto 1/x^2$ is convex on $\mathbb{R}^+$, $\max_{|r|\leq 1/2}\sum_{n\neq 0}\frac{1}{(n-r)^2} = \sum_{n\neq 0}\frac{1}{(n-1/2)^2} = \pi^2 - 4$.

Write $q = q^+q^-$, where $q^+ = \gcd(q,w)$ and $q^- = q/q^+$. We can assume q is square-free at this stage, since otherwise all terms with dm divisible by q vanish; then $\gcd(q^+,q^-) = 1$. Summing over all $m \leq M$, $d|w$ with $q|dm$, we obtain,

using (3.7), that

$$\begin{aligned}\sum_{\substack{m\le M\\ \gcd(m,w)=1}} \mu(m) \sum_{\substack{d|w\\ q|dm}} \mu(d) \sum_n e(\alpha m d n)\eta(mdn/x)\\
&= x\widehat{\eta}(-\delta) \sum_{\substack{m\le M\\ \gcd(m,w)=1\\ q^-|m}} \frac{\mu(m)}{m} \sum_{\substack{d|w\\ q^+|d}} \frac{\mu(d)}{d}\\
&+ O_{\le}\left(\left(\frac{1}{4}-\frac{1}{\pi^2}\right)\frac{|\eta''|_1}{x} \sum_{\substack{m\le M\\ q^-|m}} m \sum_{\substack{d|w\\ q^+|d}} d\right)\\
&= \frac{x\mu(q)}{q}\frac{\phi(w/q^+)}{w/q^+}\widehat{\eta}(-\delta)\cdot \sum_{\substack{m\le M/q^-\\ \gcd(m,wq^-)=1}} \frac{\mu(m)}{m}\\
&+ O_{\le}\left(\frac{M(M+q^-)q^+\sigma(w/q^+)}{xq^-}\cdot\left(\frac{1}{8}-\frac{1}{2\pi^2}\right)|\eta''|_1\right).\end{aligned}$$

Now consider the terms with md not divisible by q, or $m > M$. Here we apply (2.1) with $f(y) = \eta(my/x)$ and αm instead of α, and obtain

$$\left|\sum_n e(\alpha m n)\eta(mn/x)\right| \le \frac{1}{2}|\eta'|_1 \cdot \min\left(\frac{cx}{m}, \frac{1}{|\sin(\pi\alpha m)|}\right), \tag{3.8}$$

where $c = 2|\eta|_1/|\eta'|_1 + 1$.

Take first the case $Q/2w \ge \lfloor D\rfloor$; then $M = \lfloor D\rfloor$. In this case, all terms with md divisible by q have been removed. The contribution of the other terms is bounded from above by

$$\frac{1}{2}|\eta'|_1 \cdot \sum_{\substack{m\le Dw\\ q\nmid m}} \min\left(\frac{cx}{m}, \frac{1}{|\sin(\pi\alpha m)|}\right).$$

By Lemma 3.2, this is at most

$$\begin{aligned}&\sum_{j=0}^{\infty} \sum_{jq<m\le\min((j+1)q-1,Dw)} \min\left(\frac{cx}{m}, \frac{1}{|\sin(\pi\alpha m)|}\right)\\
&\le \frac{2q}{\pi}\log 2eq + \sum_{j=1}^{\lceil Dw/q\rceil -1} \frac{2q}{\pi}\log\frac{16cx}{jq} \le \frac{2q}{\pi}\log 2eq + \int_0^{Dw} \frac{2}{\pi}\log\frac{16cx}{t}dt.\end{aligned}$$

Let us now examine the case $Q/2w < \lfloor D\rfloor$; then $M = Q/2w$. The sum of the terms in (3.6) with md not divisible by q is bounded from above by

$$\frac{1}{2}|\eta'|_1 \cdot \left(\sum_{\substack{m\le M\\ q\nmid m}} \min\left(\frac{cx}{m}, \frac{1}{|\sin(\pi\alpha m)|}\right) + \sum_{M<m\le Dw} \min\left(\frac{cx}{m}, \frac{1}{|\sin(\pi\alpha m)|}\right)\right).$$

As above, Lemma 3.2 gives us that the sum over $m \leq M$, $q \nmid m$ is at most
$$\frac{2q}{\pi} \log 2eq + \int_0^M \frac{2}{\pi} \log \frac{16cx}{t} dt.$$

Let us now treat the terms with $M < m \leq Dw$ using the approximation a'/q' instead of a/q. Lemma 3.1 gives us

$$\begin{aligned}
&\sum_{\substack{M<m\leq Dw\\ q\nmid m}} \min\left(\frac{cx}{m}, \frac{1}{|\sin(\pi\alpha m)|}\right)\\
&\leq \sum_{j=0}^{\infty} \sum_{jq'+M<m\leq \min((j+1)q'+M,Dw)} \min\left(\frac{cx}{m}, \frac{1}{|\sin(\pi\alpha m)|}\right)\\
&\leq \sum_{j=0}^{\lceil (Dw-M)/q'\rceil -1} \left(\frac{3cx}{\lfloor M+jq'\rfloor +1} + \frac{2q'}{\pi} \log \frac{2ecx}{\lfloor M+jq'\rfloor +1}\right).
\end{aligned}$$

The first kind of terms within the sum makes a small contribution: since $q' \geq (\epsilon/(1+\epsilon))Q$ and $M = Q/2w$,

(3.9)
$$\sum_{j=0}^{\lceil (Dw-M)/q'\rceil -1} \frac{3cx}{M+jq'} \leq \frac{3cx}{M} + \frac{1}{q'} \int_M^{Dw} \frac{3cx}{t} dt = \frac{6wcx}{Q} + \frac{3(1+\epsilon)cx}{\epsilon Q} \log \frac{Dw}{M}.$$

The second kind of terms add up to
$$\sum_{j=0}^{\lceil (Dw-M)/q'\rceil -1} \frac{2q'}{\pi} \log \frac{2ecx}{\lfloor M+jq'\rfloor +1} \leq \frac{2q'}{\pi} \log \frac{4ewcx}{Q+1} + \frac{2}{\pi} \int_M^{Dw} \log \frac{2ecx}{t} dt.$$

Recall that $q' \leq Q' \leq (1+\epsilon)Q + 1 \leq (1+\epsilon)(Q+1)$. Recall as well that $Q/2 < w\lfloor D\rfloor$; since Q is an integer, it follows that $Q+1 \leq 2wD$. Since $t \mapsto t\log(4ecx/t)$ is increasing for $t \in (0, 4cx]$ and $(Q+1) \leq 2wD \leq 2x < 4cx$,

$$\frac{2q'}{\pi} \log \frac{4ewcx}{Q+1} \leq \frac{4w(1+\epsilon)D}{\pi} \log \frac{4ewcx}{2wD} = \frac{4w(1+\epsilon)D}{\pi} \log \frac{4ecx}{2D}. \tag{3.10}$$

Summing all terms with m not divisible by q, we obtain at most
$$\frac{2q}{\pi} \log 2eq + \frac{2}{\pi} \int_0^{Dw} \log \frac{16cx}{t} dt + \frac{4w(1+\epsilon)D}{\pi} \log \frac{2ecx}{D} + \frac{6wcx}{Q} + \frac{3(1+\epsilon)cx}{\epsilon Q} \log \frac{Dw}{M}$$
where the last three terms are present only if $Q/2w < \lfloor D\rfloor$.

Now
$$\frac{2}{\pi} \int_0^{Dw} \log \frac{16cx}{t} dt = \frac{2}{\pi} Dw \log \frac{16ecx}{Dw}.$$
We remember to multiply by $(1/2)|\eta'|_1$ (as in (3.8)) and obtain
$$\begin{aligned}
&|\eta_1'|\cdot \left(\frac{q}{\pi} \log 2eq + \frac{(2+2\epsilon)Dw}{\pi} \log \frac{2ecx}{D} + \frac{Dw}{\pi} \log \frac{16ecx}{wD}\right)\\
&\quad + (2|\eta_1| + |\eta_1'|) \cdot \left(\frac{3wcx}{Q} + \frac{3(1+\epsilon)x}{2\epsilon Q} \log \frac{Dw}{M}\right),
\end{aligned}$$
with the slightly better bound
$$|\eta_1'| \cdot \left(\frac{q}{\pi} \log 2eq + \frac{1}{\pi} Dw \log \frac{16ecx}{Dw}\right)$$

if $Q/2w \geq \lfloor D \rfloor$. □

We will also need a version of Lemma 3.3 with a factor of $(\log n)$.

Lemma 3.4. *Let $\alpha = a/q + \delta/x$, $\gcd(a,q) = 1$, $|\delta/x| \leq 1/q^2$. Let η be continuous, piecewise C^2 and compactly supported, with $\eta'' \in L_1$. Let $w \in \mathbb{Z}^+$ be square-free. Assume that, for any $\rho \geq \rho_0$, ρ_0 a constant, the function $\eta_{1,\rho}(t) = \log(\rho t)\eta_0(t)$ satisfies*

$$|\eta_{1,\rho}|_1 \leq \log(\rho)|\eta|_1, \quad |\eta'_{1,\rho}|_1 \leq \log(\rho)|\eta'|_1, \quad |\eta''_{1,\rho}|_1 \leq \log(\rho)|\eta''|_1. \tag{3.11}$$

Then, for $D \leq x/\max(ew, \rho_0, 4.33)$ and any $\epsilon \in (0,1]$,

$$\begin{aligned}
&\sum_{\substack{m \leq D \\ \gcd(m,w)=1}} \mu(m) \sum_{\substack{n \\ \gcd(n,w)=1}} (\log n) e(\alpha m n)\eta(mn/x) \\
&= \frac{x\mu(q)}{q}\frac{\phi(w/q^+)}{w/q^+}\widehat{\eta}(-\delta) \cdot \sum_{\substack{m \leq M/q^- \\ \gcd(m,wq^-)=1}} \frac{\mu(m)}{m}\log\left(\frac{x}{mq}\right) \\
&+ \frac{x\mu(q)}{q}\left(\frac{\phi(w/q^+)}{w/q^+}\widehat{\log\cdot\eta}(-\delta) - \left(\sum_{d|\frac{w}{q^+}} \frac{\mu(d)\log d}{d}\right)\widehat{\eta}(-\delta)\right) \sum_{\substack{m \leq M/q^- \\ \gcd(m,wq^-)=1}} \frac{\mu(m)}{m} \\
&+ O_{\leq}\left(\left(\frac{q}{\pi}\log 2eq \log x\right)|\eta'|_1\right) \\
&+ O_{\leq}\left(\left(\frac{2\tau(w)}{e} + \sigma(w)\frac{D^2 w}{xq}\log\frac{e^{1/2}x}{D}\right)\cdot\left(\frac{1}{8} - \frac{1}{2\pi^2}\right)|\eta''|_1\right) \\
&+ O_{\leq}\left(\left(\frac{3wcx}{Q} + \frac{3(1+\epsilon)cx}{2\epsilon Q}\log\frac{Dw}{M}\right)\left(\log\frac{x}{M}\right)|\eta'|_1\right) \\
&+ O_{\leq}\left(\left(\frac{3+2\epsilon}{\pi}Dw\log\frac{4ecx}{D}\log\frac{ex}{D}\right)|\eta'|_1\right),
\end{aligned} \tag{3.12}$$

where $c = 2|\eta|_1/|\eta'|_1 + 1$, $Q = \lfloor x/|\delta q|\rfloor$, $M = \min(\lfloor D \rfloor, Q/2w)$, $q^+ = \gcd(q,w)$ and $q^- = q/q^+$. If $Q/2w > D$, the last two lines of (3.12) are replaced by

$$\frac{|\eta'|_1}{\pi}Dw\left(\log\frac{16cex}{Dw}\log\frac{ex}{Dw} + 1\right). \tag{3.13}$$

Proof. Define Q, Q', a', q' and M as in the proof of Lemma 3.3. The same proof works as for Lemma 3.3; we go over the differences. When applying Poisson summation or (2.1), use $\eta_{1,x/md}(t) = (\log xt/md)\eta(t)$ instead of $\eta(t)$. Then use (3.11) to bound the ℓ_1 norms of $\eta_{1,x/md}$ and its derivatives; for example,

$$|\eta''_{1,x/md}|_1 \leq \log\frac{x}{md}|\eta''|_1.$$

Then, for $f(u) = e((\delta m/x)u)(\log u)\eta((m/x)u)$,

$$\begin{aligned}
\sum_n \widehat{f}(n) &= \frac{x}{md}\widehat{\eta_{1,x/md}}(-\delta) + \frac{md}{x}\frac{|\eta''_{1,x/md}|_1}{(2\pi)^2}O_{\leq}(\pi^2 - 4) \\
&= \frac{x}{md}\left(\widehat{\log\cdot\eta}(-\delta) + \log\left(\frac{x}{md}\right)\widehat{\eta}(-\delta)\right) + \frac{md}{x}\log\left(\frac{x}{md}\right)O_{\leq}\left(\frac{|\eta''|_1}{4} - \frac{|\eta''|_1}{\pi^2}\right)
\end{aligned}$$

The part of the main term involving $\log(x/md)$ becomes

$$\begin{aligned} x\widehat{\eta}(-\delta) &\sum_{\substack{m\leq M\\ \gcd(m,w)=1\\ q^-|m}} \frac{\mu(m)}{m} \sum_{\substack{d|w\\ q^+|d}} \frac{\mu(d)}{d} \log\left(\frac{x}{md}\right)\\ &= \frac{x\mu(q)}{q}\frac{\phi(w/q^+)}{w/q^+}\widehat{\eta}(-\delta)\cdot \sum_{\substack{m\leq M/q^-\\ \gcd(m,wq^-)=1}} \frac{\mu(m)}{m}\log\left(\frac{x}{mq}\right)\\ &- \frac{x\mu(q)}{q}\widehat{\eta}(-\delta)\cdot \sum_{\substack{m\leq M/q^-\\ \gcd(m,wq^-)=1}} \frac{\mu(m)}{m} \sum_{d|w/q^+} \frac{\mu(d)}{d}\log d \end{aligned}$$

It remains to see how the same factor affects the error terms.

In the term in front of $(|\eta''|_1/4 - |\eta''|_1/\pi^2)$, we find the sum

$$\sum_{\substack{m\leq M\\ q^-|m}} m\log\left(\frac{x}{md}\right) \leq q^-\left(\frac{x}{edq^-} + \int_0^{M/q^-} t\log\frac{x/(q^-d)}{t}dt\right) = \frac{x}{de} + \frac{M^2}{2q^-}\log\frac{e^{1/2}x}{dM},$$

where we use the fact that $t\mapsto t\log(x/t)$ is increasing for $t\leq x/e$. By the same fact (and by $M\leq D$), $(M^2/q^-)\log(e^{1/2}x/dM) \leq (D^2/q^-)\log(e^{1/2}x/dD)$.

The basic estimate for the rest of the proof (replacing (3.8)) is

$$\begin{aligned} \sum_n e(\alpha mn)(\log n)\eta\left(\frac{mn}{x}\right) &= \sum_n e(\alpha mn)\eta_{1,x/m}\left(\frac{mn}{x}\right)\\ &\leq \frac{1}{2}|\eta'|_{1,x/m}\cdot\min\left(\frac{cx}{m}, \frac{1}{|\sin(\pi\alpha m)|}\right)\\ &\leq \frac{\log x/m}{2}|\eta'|_1\cdot\min\left(\frac{cx}{m}, \frac{1}{|\sin(\pi\alpha m)|}\right). \end{aligned}$$

The term $(q/\pi)\log 2eq$ becomes $(q/\pi)\log x\log 2eq$; the terms in (3.9) get multiplied by $\log x/M$. Recall that $(Q+1)\leq 2wD$. The analogue of (3.10) is

$$\frac{2q'}{\pi}\log\frac{4ewcx}{Q+1}\log\frac{2wx}{Q+1} \leq \frac{4(1+\epsilon)wD}{\pi}\log\frac{4ecx}{2D}\log\frac{2x}{2D},$$

which holds because $D < x/4.33$ and $c\geq 1$ imply that we are in the range where the function $t\mapsto t\log(4ewcx/t)\log(2wx/t)$ is increasing. The other term of this type is the one coming from the integral

$$\frac{2}{\pi}\int_0^{Dw}\log\frac{16cx}{t}\log\frac{x}{t}dt = \frac{2}{\pi}Dw\left(\log\frac{16cex}{Dw}\log\frac{ex}{Dw}+1\right).$$

The two terms, summed together, give less than

$$\frac{6+4\epsilon}{\pi}Dw\log\frac{4ecx}{D}\log\frac{ex}{D},$$

which is then multiplied by $(1/2)|\eta'|_1$. □

We will apply the following only for q relatively large.

Lemma 3.5. *Let $\alpha = a/q + O_{\leq}(1/q^2)$, $\gcd(a,q) = 1$, $q > 2e^4$. Let η be continuous, piecewise C^2 and compactly supported, with $\eta'' \in L_1$. Let $w \in \mathbb{Z}^+$ be square-free. Then, for any $r \geq 1$, $U, V \geq 1$ with $UV \leq x/w$ and $\epsilon \leq 1$,*

$$\begin{aligned}
&\left|\sum_{\substack{v\leq V\\ \gcd(v,w)=1}} \Lambda(v) \sum_{\substack{u\leq U\\ \gcd(u,w)=1}} \mu(u) \sum_{\substack{n\\ \gcd(n,w)=1}} e(\alpha vun)\eta(vun/x)\right| \\
&\leq \frac{x}{q}|\widehat{\eta}(\delta)|\frac{\phi(w/q^+)}{w/q^+}\log(Vq^-) + (\pi^2-4)\cdot\frac{|\eta''|_1}{(2\pi)^2}\frac{q^+\sigma(w/q^+)}{x} \\
&\cdot\left(1.03883\frac{UV^2}{2} + \frac{(\log V)(UV)^2}{2q^-} + \frac{U^2+U}{2}(\log q^-)q^-\right) \\
&+ \frac{1}{2}|\eta'|_1\cdot\left(\frac{2q}{\pi}\log 2eq\log q + \frac{2}{\pi}UV\left(\log\frac{UV}{e}\log\frac{16ecx}{UV} - 1\right)\right) \\
&+ \frac{1}{2}|\eta'|_1\cdot\frac{4q}{\pi}\log(UV+q)\log 16cx,
\end{aligned} \tag{3.14}$$

if $UV < Q/2w$, where $c = 2|\eta|_1/|\eta'|_1 + 1$, $Q = \lfloor x/|\delta q|\rfloor$, $q^+ = \gcd(q,w)$ and $q^- = q/q^+$. If $UV \geq Q/2w$,

$$\begin{aligned}
&\left|\sum_{\substack{v\leq V\\ \gcd(v,w)=1}} \Lambda(v) \sum_{\substack{u\leq U\\ \gcd(u,w)=1}} \mu(u) \sum_{\substack{n\\ \gcd(n,w)=1}} e(\alpha vun)\eta(vun/x)\right| \\
&\leq \frac{x|\widehat{\eta}(\delta)|}{q}\frac{\phi(w/q^+)}{w/q^+}\log(Vq^-) \\
&+ \frac{1}{2}|\eta'|_1\cdot\frac{2}{\pi}UVw\left(\log\frac{UVw}{e}\log\frac{16ecx}{UVw} - 1\right) \\
&+ \frac{1}{2}|\eta'|_1\cdot\frac{4}{\pi}(1+\epsilon)UVw\left(\log UVw\cdot\log\frac{2ecx}{UV} + \log 2ecx\cdot\max(\log w, 1)\right) \\
&+ \frac{1}{2}|\eta'|_1\cdot\left(\frac{4q}{\pi}\log(UV+q)\log 16cx + \frac{2q}{\pi}\log 2eq\log q\right) \\
&+ (\pi^2-4)\frac{|\eta''|_1}{(2\pi)^2}\frac{\sigma(w)}{x}\left(1.03883\frac{QV}{4w} + \frac{\log V}{2q^-}\left(\frac{Q}{2w}\right)^2 + \frac{U^2+U}{2}(\log q^-)q^-\right) \\
&+ \frac{1}{2}|\eta'|_1\cdot\left(\frac{6wcx}{Q} + \frac{3(1+\epsilon)cx}{\epsilon Q}\log\frac{2UVw^2}{Q}\right)\log UVw.
\end{aligned} \tag{3.15}$$

Proof. We proceed essentially as in Lemma 3.3; let Q, q', Q', M, q^+ and q^- be as there. The sum can be written as

$$\sum_{\substack{v\leq V\\ \gcd(v,w)=1}} \Lambda(v) \sum_{\substack{u\leq U\\ \gcd(u,w)=1}} \mu(u)\sum_{d|w}\mu(d)\sum_{n} e(\alpha vudn)\eta(vudn/x).$$

Let $M = \min(\lfloor UV \rfloor, Q/2w)$. The terms with $uv \leq M$, uvd divisible by q contribute

(3.16)
$$\sum_{\substack{a\leq M\\ \gcd(a,w)=1\\ q^-|a}} \left(\sum_{\substack{v|a\\ a/U\leq v\leq V}} \Lambda(v)\mu(a/v)\right) \sum_{\substack{d|w\\ q^+|d}} \left(\mu(d)\frac{x\widehat{\eta}(\delta)}{ad} + O\left(\frac{ad}{x}\frac{|\eta''|_1}{(2\pi)^2}\cdot(\pi^2-4)\right)\right).$$

The contribution of $x\widehat{\eta}(\delta)/ad$ can be rewritten as

(3.17)
$$\begin{aligned} &x\widehat{\eta}(\delta)\frac{\phi(w/q^+)}{w/q^+}\left(\frac{\mu(q^+)\mu(q^-)}{q}\cdot \sum_{\substack{v\leq V\\ \gcd(v,wq^-)=1}} \frac{\Lambda(v)}{v} \sum_{\substack{u\leq \min(U/q^-, M/Vq^-)\\ \gcd(u,wq^-)=1}} \frac{\mu(u)}{u}\right)\\ &+ x\widehat{\eta}(\delta)\frac{\phi(w/q^+)\mu(q^+)}{w/q^+}\cdot\frac{1}{q}\sum_{\substack{p^\alpha\leq V\\ p^\alpha|q^-}} \mu\left(\frac{q^-}{p^\alpha}\right)(\log p) \sum_{\substack{u\leq \min(U/(q^-/p^\alpha), M/(Vq^-/p^\alpha))\\ \gcd(u,wq^-/p^\alpha)=1}} \frac{\mu(u)}{u}.\end{aligned}$$

We are happy to bound the innermost sum by 1 (using (2.5)), and obtain a total of at most

$$\frac{x|\widehat{\eta}(\delta)|}{q}\frac{\phi(w/q^+)}{w/q^+}\left(\log V + \log q^-\right) \tag{3.18}$$

by (2.10); recall also that $\widehat{\eta}(\delta) \leq \min(|\eta|_1, |\eta''|_1/(2\pi\delta)^2)$ (see (2.2)).

The second term in (3.16) contributes

$$\begin{aligned}(\pi^2-4)\frac{|\eta''|_1}{(2\pi)^2}\frac{1}{x} &\sum_{\substack{u\leq U\\ \gcd(uv,w)=1\\ uv\leq M,\ q^-|uv\\ u \text{ sq-free}}}\sum_{v\leq V} \Lambda(v)\sum_{\substack{d|w\\ q^+|d}} uvd.\\ &= (\pi^2-4)\frac{|\eta''|_1}{(2\pi)^2}\frac{q^+\sigma(w/q^+)}{x} \sum_{\substack{u\leq U\\ \gcd(uv,w)=1\\ uv\leq M,\ q^-|uv\\ u \text{ sq-free}}}\sum_{v\leq V} \Lambda(v)uv.\end{aligned}$$

By the estimates (2.10) and (2.12), the double sum here is at most

(3.19)
$$\begin{aligned}&\sum_{\substack{v\leq V\\ \gcd(v,q^-)=1}} \Lambda(v)v\frac{\min(U,M/v)(\min(U,M/v)+q^-)}{2q^-} + \sum_{\substack{p\leq V\\ p|q^-}}(\log p)p\frac{U(U+q^-/p)}{2q^-/p}\\ &\leq 1.03883\frac{MV}{2} + \frac{1}{2q^-}\log(V)M^2 + \frac{U^2+U}{2}(\log q^-)q^-\end{aligned}$$

(A stronger bound, without the first term, holds for q non-square-free.)

From this point onwards, we use the easy bound

$$\left|\sum_{\substack{v|a\\ a/U\le v\le V}} \Lambda(v)\mu(a/v)\right| \le \log a.$$

What we must bound now is

$$\sum_{\substack{m\le UV\\ \gcd(m,w)=1\\ q\nmid m \text{ or } m>M}} (\log m) \sum_{\substack{n\\ \gcd(n,w)=1}} e(\alpha mn)\eta(mn/x).$$

Now use (3.8) and Lemmas 3.1–3.2. The terms with $m \le M$, $q \nmid m$ contribute at most

$$\frac{1}{2}|\eta'|_1 \cdot \left(\frac{2q}{\pi}\log 2eq\log q + \sum_{j=1}^{\lceil M/q\rceil-1} (\log(j+1)q)\cdot\frac{2q}{\pi}\log\frac{16cx}{jq}\right). \tag{3.20}$$

For any $t \in [a,b]$, $0 < a \le b \le y$,

$$\begin{aligned}\left|\log b\log\frac{y}{a} - \log t\log\frac{y}{t}\right| &= \left|\log\frac{b}{t}\log\frac{y}{a} + \log t\log\frac{t}{a}\right|\\ &\le \log\max\left(\frac{y}{a},t\right)\cdot\log\frac{b}{a} \le \log y\cdot\log\frac{b}{a}.\end{aligned} \tag{3.21}$$

Hence (taking out the term with $j = \lceil M/q\rceil - 1$) we see that the sum in (3.20) is at most

(3.22)

$$\begin{aligned}&\frac{2}{\pi}\int_0^M \log t\log\frac{16cx}{t}dt + \frac{2q}{\pi}\log(M+q)\log\frac{16cx}{q} + \frac{2q}{\pi}\sum_{j=1}^{\lceil M/q\rceil-1}\log\frac{j+1}{j}\log 16cx\\ &\le \frac{2M}{\pi}\left(\log\frac{M}{e}\log\frac{16ecx}{M} - 1\right) + \frac{4q}{\pi}\log(M+q)\log 16cx.\end{aligned}$$

If $\lfloor UV\rfloor \le Q/2w$, then we have considered all terms, and get a total of at most

$$\begin{aligned}&\frac{x}{q}|\widehat{\eta}(\delta)|\frac{\phi(w/q^+)}{w/q^+}\log(Vq^-) + (\pi^2-4)\cdot\frac{|\eta''|_1}{(2\pi)^2}\frac{q^+\sigma(w/q^+)}{x}\\ &\quad\cdot\left(1.03883\frac{UV^2}{2} + \frac{(\log V)(UV)^2}{2q^-} + \frac{U^2+U}{2}(\log q^-)q^-\right)\\ &\quad+\frac{1}{2}|\eta'|_1\cdot\left(\frac{2q}{\pi}\log 2eq\log q + \frac{2}{\pi}UV\left(\log\frac{UV}{e}\log\frac{16ecx}{UV} - 1\right)\right)\\ &\quad+\frac{1}{2}|\eta'|_1\cdot\frac{4q}{\pi}\log(UV+q)\log 16cx,\end{aligned}$$

since, in that case, $M = \lfloor UV\rfloor$.

Assume from now on that $\lfloor UV\rfloor > Q/2w$. Then $M = Q/2w$. The terms with $m \le M$, $q \nmid m$ contribute (3.20) as before, whereas the terms with $m > M$

contribute at most

$$\frac{1}{2}|\eta'|_1 \cdot \sum_{j=0}^{\lceil (Dw-M)/q'\rceil -1} \left(\frac{3cx}{\lfloor M\rfloor + jq'+1} + \frac{2q'}{\pi}\log\frac{2ecx}{\lfloor M\rfloor + jq'+1}\right) \cdot \log\min(\lfloor M\rfloor + (j+1)q', Dw), \tag{3.23}$$

where $D = UV$. As in the proof of Lemma 3.3, $q' \geq (\epsilon/(1+\epsilon))Q$.

The first kind of term within the sum in (3.23) adds up to at most

$$\log Dw \cdot \sum_{j=0}^{\left\lceil \frac{Dw-M}{q'}\right\rceil -1} \frac{3cx}{M+jq'} \leq \log Dw \cdot \left(\frac{6wcx}{Q} + \frac{3(1+\epsilon)cx}{\epsilon Q}\log\frac{Dw}{M}\right), \tag{3.24}$$

much as in (3.9). We wish to approximate

$$\begin{aligned}&\frac{2q}{\pi}\sum_{j=1}^{\lceil M/q\rceil -1} (\log(j+1)q)\log\frac{16cx}{jq}\\ &+\frac{2q'}{\pi}\sum_{j=0}^{\lceil (Dw-M)/q'\rceil -1} (\log\min(\lfloor M\rfloor + (j+1)q', Dw))\log\frac{2ecx}{\lfloor M\rfloor + jq'+1}\end{aligned} \tag{3.25}$$

by an integral. By (3.21), the first sum is (as in (3.22)) at most

$$\frac{2}{\pi}\int_0^M \log t\log\frac{16cx}{t}dt + \frac{4q}{\pi}\log(M+q)\log 16cx.$$

The second sum, again by (3.21), is at most

$$\begin{aligned}&\frac{2}{\pi}\int_M^{Dw}\log t\log\frac{2ecx}{t}dt + \frac{2q'}{\pi}\log Dw\log\frac{2ecx}{\lfloor M\rfloor+1}\\ &+\frac{2q'}{\pi}\sum_{j=0}^{\lceil (Dw-M)/q'\rceil -2}\log\frac{\lfloor M\rfloor + (j+1)q'+1}{\lfloor M\rfloor + jq'+1}\cdot\log 2ecx.\end{aligned}$$

Clearly

$$\begin{aligned}\int_0^M \log t\log\frac{16cx}{t}dt + \int_M^{Dw}\log t\log\frac{2ecx}{t}dt &= \int_0^{Dw}\log t\log\frac{16cx}{t}dt\\ &\leq Dw\left(\log\frac{Dw}{e}\log\frac{16ecx}{Dw} - 1\right).\end{aligned}$$

We conclude that (3.25) is at most

$$\begin{aligned}&\frac{2}{\pi}Dw\left(\log\frac{Dw}{e}\log\frac{16ecx}{Dw} - 1\right) + \frac{4q}{\pi}\log(M+q)\log 16cx\\ &+\frac{2q'}{\pi}\log Dw\log\frac{2ecx}{\lfloor M\rfloor+1} + \frac{2q'}{\pi}\log\frac{Dw}{\lfloor M\rfloor+1}\log 2ecx.\end{aligned} \tag{3.26}$$

Note that (much as in the proof of Lemma 3.3) $q' \leq (1+\epsilon)(Q+1)$, $Q < 2\lfloor D\rfloor w$ (implying $Q+1 \leq 2\lfloor D\rfloor w \leq 2Dw$) and $M = Q/2w$ (implying $\lfloor M\rfloor + 1 = \lfloor Q/2w\rfloor + 1 \geq (Q+1)/2w$ and $M < D = UV$). Proceeding as before (3.10), we

obtain that (3.26) is at most

$$\frac{2}{\pi} Dw \left(\log \frac{Dw}{e} \log \frac{16ecx}{Dw} - 1\right) + \frac{4q}{\pi} \log(UV + q) \log 16cx$$
$$+ \frac{4Dw}{\pi}(1+\epsilon)\left(\log Dw \log \frac{2ecx}{D} + \log 2ecx \max(\log w, 1)\right).$$

□

4. Type II

We must now consider the sum

$$\sum_{\substack{m>U \\ \gcd(m,w)=1}} \left(\sum_{\substack{d>U \\ d|m}} \mu(d)\right) \sum_{\substack{n>V \\ \gcd(n,w)=1}} \Lambda(n) e(\alpha m n) \eta(mn/x). \tag{4.1}$$

Here the main improvements over classical treatments are as follows:

(1) obtaining cancellation in the term $\mu_{>U} * 1$, leading to a gain of a factor of log;
(2) using a large sieve for primes, getting rid of a further log;
(3) exploiting, via a non-conventional application of the principle of the large sieve (Lemma 4.3), the fact that α is in the tail of an interval (when that is the case).

Some of the techniques developed for (1) should be applicable to other instances of Vaughan's identity in the literature.

It is technically helpful to express η as the (multiplicative) convolution of two functions of compact support:

$$\eta(x) = \int_0^\infty \eta_1(t)\eta_2(x/t)\frac{dt}{t}. \tag{4.2}$$

For the smoothing function $\eta(t) = \eta_0(t) = 4\max(\log 2 - |\log 2t|, 0)$, (4.2) holds with

$$\eta_1(t) = \eta_2(t) = \begin{cases} 2 & \text{if } t \in (1/2, 1] \\ 0 & \text{otherwise.} \end{cases} \tag{4.3}$$

We will work with $\eta_1(t)$, $\eta_2(t)$ as in (4.3) for convenience, yet what follows should carry over to other (non-negative) choices of η_1 and η_2.

By (4.2), the sum (4.1) equals

(4.4)

$$4\int_0^\infty \sum_{\substack{m>U \\ \gcd(m,w)=1}} \left(\sum_{\substack{d>U \\ d|m}} \mu(d)\right) \sum_{\substack{n>V \\ \gcd(n,w)=1}} \Lambda(n) e(\alpha m n) \eta_1(t) \eta_2\left(\frac{mn/x}{t}\right) \frac{dt}{t}$$
$$= 4\int_V^{x/U} \sum_{\substack{\max\left(\frac{x}{2W}, U\right) < m \leq \frac{x}{W} \\ \gcd(m,w)=1}} \left(\sum_{\substack{d>U \\ d|m}} \mu(d)\right) \sum_{\substack{\max\left(V, \frac{W}{2}\right) < n \leq W \\ \gcd(n,w)=1}} \Lambda(n) e(\alpha m n) \frac{dW}{W}$$

by the substitution $t = (m/x)W$. (We can assume $V \leq W \leq x/U$ because otherwise one of the sums in (4.5) is empty.)

We separate n prime and n non-prime. By Cauchy-Schwarz, the expression within the integral in (4.4) is then at most $\sqrt{S_1(U,W)\cdot S_2(U,V,W)} + \sqrt{S_1(U,W)\cdot S_3(W)}$, where

$$S_1(U,W) = \sum_{\substack{\max\left(\frac{x}{2W},U\right)<m\leq\frac{x}{W}\\ \gcd(m,w)=1}} \left(\sum_{\substack{d>U\\ d|m}} \mu(d)\right)^2,$$

(4.5)

$$S_2(U,V,W) = \sum_{\substack{\max\left(\frac{x}{2W},U\right)<m\leq\frac{x}{W}\\ \gcd(m,w)=1}} \left|\sum_{\substack{\max\left(V,\frac{W}{2}\right)<p\leq W\\ \gcd(p,w)=1}} (\log p) e(\alpha m p)\right|^2.$$

and

(4.6)
$$\begin{aligned} S_3(W) &= \sum_{\substack{\frac{x}{2W}<m\leq\frac{x}{W}\\ \gcd(m,w)=1}} \left|\sum_{\substack{n\leq W\\ n \text{ non-prime}}} \Lambda(n)\right|^2 \\ &= \sum_{\substack{\frac{x}{2W}<m\leq\frac{x}{W}\\ \gcd(m,w)=1}} \left(1.42620 W^{1/2}\right)^2 \leq 1.0171x + 2.0341W \end{aligned}$$

(by [RS62, Thm. 13]). We will assume $V \leq w$; thus the condition $\gcd(p,w)=1$ will be fulfilled automatically and can be removed.

The contribution of $S_3(W)$ will be negligible. We must bound $S_1(U,W)$ and $S_2(U,V,W)$ from above.

4.1. **The sum S_1: cancellation.** We shall bound

$$S_1(U,W) = \sum_{\substack{\max(U,x/2W)<m\leq x/W\\ \gcd(m,w)=1}} \left(\sum_{\substack{d>U\\ d|m}} \mu(d)\right)^2.$$

There will be what is perhaps a surprising amount of cancellation: the expression within the sum will be bounded by a constant on average.

4.1.1. *Reduction to a sum with μ.* We can write

(4.7)
$$\begin{aligned} \sum_{\substack{\max(U,x/2W)<m\leq x/W\\ \gcd(m,w)=1}} \left(\sum_{\substack{d>U\\ d|m}} \mu(d)\right)^2 &= \sum_{\substack{\frac{x}{2W}<m\leq\frac{x}{W}\\ \gcd(m,w)=1}} \sum_{d_1,d_2|m} \mu(d_1>U)\mu(d_2>U) \\ &= \sum_{\substack{r_1<x/WU\\ \gcd(r_1,r_2)=1\\ \gcd(r_1r_2,w)=1}} \sum_{r_2<x/WU} \sum_{\substack{l\\ \gcd(l,r_1r_2)=1\\ r_1l,r_2l>U\\ \gcd(\ell,w)=1}} \mu(r_1 l)\mu(r_2 l) \sum_{\substack{\frac{x}{2W}<m\leq\frac{x}{W}\\ r_1r_2l|m\\ \gcd(m,w)=1}} 1, \end{aligned}$$

where we write $d_1 = r_1 l$, $d_2 = r_2 l$, $l = \gcd(d_1, d_2)$. (The inequality $r_1 < x/WU$ comes from $r_1 r_2 l | m$, $m \leq x/W$, $r_2 l > U$; $r_2 < x/WU$ is proven in the same way.) Now (4.7) equals

$$\sum_{\substack{s < \frac{x}{WU} \\ \gcd(s,w)=1}} \sum_{\substack{r_1 < \frac{x}{WUs} \\ \gcd(r_1,r_2)=1 \\ \gcd(r_1 r_2, w)=1}} \sum_{r_2 < \frac{x}{WUs}} \mu(r_1)\mu(r_2) \sum_{\substack{\max\left(\frac{U}{\min(r_1,r_2)}, \frac{x/W}{2 r_1 r_2 s}\right) < l \leq \frac{x/W}{r_1 r_2 s} \\ \gcd(l, r_1 r_2)=1, (\mu(l))^2 = 1 \\ \gcd(\ell, w)=1}} 1, \tag{4.8}$$

where we have set $s = m/(r_1 r_2 l)$.

Lemma 4.1. *Let $z, y > 0$. Then*

$$\sum_{\substack{r_1 < y \\ \gcd(r_1,r_2)=1 \\ \gcd(r_1 r_2, w)=1}} \sum_{r_2 < y} \mu(r_1)\mu(r_2) \sum_{\substack{\min\left(\frac{z/y}{\min(r_1,r_2)}, \frac{z}{2 r_1 r_2}\right) < l \leq \frac{z}{r_1 r_2} \\ \gcd(l, r_1 r_2)=1, (\mu(l))^2 = 1 \\ \gcd(\ell, w)=1}} 1 \tag{4.9}$$

equals

$$\begin{aligned} &\frac{6z}{\pi^2} \frac{w}{\sigma(w)} \sum_{\substack{r_1 < y \\ \gcd(r_1,r_2)=1 \\ \gcd(r_1 r_2, w)=1}} \sum_{r_2 < y} \frac{\mu(r_1)\mu(r_2)}{\sigma(r_1)\sigma(r_2)} \left(1 - \max\left(\frac{1}{2}, \frac{r_1}{y}, \frac{r_2}{y}\right)\right) \\ &+ O_{\leq}\left(5.08\, \zeta\left(\frac{3}{2}\right)^2 y\sqrt{z} \cdot \prod_{p|w} \left(1 + \frac{1}{\sqrt{p}}\right)\left(1 - \frac{1}{p^{3/2}}\right)^2\right). \end{aligned} \tag{4.10}$$

If $w = 2$, the error term in (4.10) can be replaced by

$$O_{\leq}\left(1.27 \zeta\left(\frac{3}{2}\right)^2 y\sqrt{z} \cdot \left(1 + \frac{1}{\sqrt{2}}\right)\left(1 - \frac{1}{2^{3/2}}\right)^2\right). \tag{4.11}$$

Proof. By Möbius inversion, (4.9) equals

$$\begin{aligned} &\sum_{\substack{r_1 < y \\ \gcd(r_1,r_2)=1 \\ \gcd(r_1 r_2, w)=1}} \sum_{r_2 < y} \mu(r_1)\mu(r_2) \sum_{\substack{l \leq \frac{z}{r_1 r_2} \\ l > \min\left(\frac{z/y}{\min(r_1,r_2)}, \frac{z}{2 r_1 r_2}\right) \\ \gcd(\ell, w)=1}} \sum_{\substack{d_1 | r_1, d_2 | r_2 \\ d_1 d_2 | l}} \mu(d_1)\mu(d_2) \\ &\qquad \sum_{\substack{d_3 | w \\ d_3 | l}} \mu(d_3) \sum_{\substack{m^2 | l \\ \gcd(m, r_1 r_2 w)=1}} \mu(m). \end{aligned} \tag{4.12}$$

We can change the order of summation of r_i and d_i by defining $s_i = r_i/d_i$, and we can also use the obvious fact that the number of integers in an interval $(a, b]$

divisible by d is $(b-a)/d + O_{\leq}(1)$. Thus (4.12) equals
(4.13)

$$\sum_{\substack{d_1,d_2<y\\ \gcd(d_1,d_2)=1\\ \gcd(d_1d_2,w)=1}} \mu(d_1)\mu(d_2) \sum_{\substack{s_1<y/d_1\\ s_2<y/d_2\\ \gcd(d_1s_1,d_2s_2)=1\\ \gcd(s_1s_2,w)=1}} \mu(d_1s_1)\mu(d_2s_2)$$

$$\sum_{d_3|w} \mu(d_3) \sum_{\substack{m\leq \sqrt{\frac{z}{d_1^2 s_1 d_2^2 s_2 d_3}}\\ \gcd(m,d_1s_1d_2s_2w)=1}} \frac{\mu(m)}{d_1d_2d_3m^2}\frac{z}{s_1d_1s_2d_2}\left(1-\max\left(\frac{1}{2},\frac{s_1d_1}{y},\frac{s_2d_2}{y}\right)\right)$$

plus

$$O_{\leq}\left(\sum_{\substack{d_1,d_2<y\\ \gcd(d_1d_2,w)=1}} \sum_{\substack{s_1<y/d_1\\ s_2<y/d_2\\ \gcd(s_1s_2,w)=1}} \sum_{d_3|w} \sum_{\substack{m\leq \sqrt{\frac{z}{d_1^2 s_1 d_2^2 s_2 d_3}}\\ m \text{ sq-free}}} 1\right). \tag{4.14}$$

If we complete the innermost sum in (4.13) by removing the condition $m \leq \sqrt{z/(d_1^2 s d_2^2 s_2)}$, we obtain (reintroducing the variables $r_i = d_i s_i$)

$$\begin{aligned} &z\cdot \sum_{\substack{r_1,r_2<y\\ \gcd(r_1,r_2)=1\\ \gcd(r_1r_2,w)=1}} \frac{\mu(r_1)\mu(r_2)}{r_1r_2}\left(1-\max\left(\frac{1}{2},\frac{r_1}{y},\frac{r_2}{y}\right)\right)\\ &\qquad\qquad \sum_{\substack{d_1|r_1\\ d_2|r_2}} \sum_{d_3|w} \sum_{\substack{m\\ \gcd(m,r_1r_2w)=1}} \frac{\mu(d_1)\mu(d_2)\mu(m)\mu(d_3)}{d_1d_2d_3m^2}\end{aligned} \tag{4.15}$$

times z. Now (4.15) equals

$$\begin{aligned}&\sum_{\substack{r_1,r_2<y\\ \gcd(r_1,r_2)=1\\ \gcd(r_1r_2,w)=1}} \frac{\mu(r_1)\mu(r_2)z}{r_1r_2}\left(1-\max\left(\frac{1}{2},\frac{r_1}{y},\frac{r_2}{y}\right)\right)\prod_{p|r_1r_2w}\left(1-\frac{1}{p}\right)\prod_{\substack{p\nmid r_1r_2\\ p\nmid w}}\left(1-\frac{1}{p^2}\right)\\ &= \frac{6z}{\pi^2}\frac{w}{\sigma(w)} \sum_{\substack{r_1,r_2<y\\ \gcd(r_1,r_2)=1\\ \gcd(r_1r_2,w)=1}} \frac{\mu(r_1)\mu(r_2)}{\sigma(r_1)\sigma(r_2)}\left(1-\max\left(\frac{1}{2},\frac{r_1}{y},\frac{r_2}{y}\right)\right),\end{aligned}$$

i.e., the main term in (4.10). It remains to estimate the terms used to complete the sum; their total is, by definition, given exactly by (4.13) with the inequality $m \leq \sqrt{z/(d_1^2 s d_2^2 s_2 d_3)}$ changed to $m > \sqrt{z/(d_1^2 s d_2^2 s_2 d_3)}$. This is a total of size at most

$$\frac{1}{2} \sum_{\substack{d_1,d_2<y\\ \gcd(d_1d_2,w)=1}} \sum_{\substack{s_1<y/d_1\\ s_2<y/d_2\\ \gcd(s_1s_2,w)=1}} \sum_{d_3|w} \sum_{\substack{m> \sqrt{\frac{z}{d_1^2 s_1 d_2^2 s_2 d_3}}\\ m \text{ sq-free}}} \frac{1}{d_1d_2d_3m^2}\frac{z}{s_1d_1s_2d_2}. \tag{4.16}$$

Adding this to (4.14), we obtain, as our total error term,

$$\sum_{\substack{d_1,d_2<y\\ \gcd(d_1d_2,w)=1}} \sum_{\substack{s_1<y/d_1\\ s_2<y/d_2\\ \gcd(s_1s_2,w)=1}} \sum_{d_3|w} f\left(\sqrt{\frac{z}{d_1^2 s_1 d_2^2 s_2 d_3}}\right), \tag{4.17}$$

where

$$f(x) := \sum_{\substack{m\leq x\\ m \text{ sq-free}}} 1 + \frac{1}{2} \sum_{\substack{m>x\\ m \text{ sq-free}}} \frac{x^2}{m^2}.$$

It is easy to see that $f(x)/x$ has a local maximum exactly when x is a square-free (positive) integer. We can hence check that

$$f(x) \leq \frac{1}{2}\left(2 + 2\left(\frac{\zeta(2)}{\zeta 4} - 1.25\right)\right) x = 1.26981\ldots x$$

for all $x \geq 0$ by checking all integers smaller than a constant and using $\{m : m \text{ sq-free}\} \subset \{m : 4 \nmid m\}$ and $1.5 \cdot (3/4) < 1.26981$ to bound f from below for x larger than a constant. Therefore, (4.17) is at most

$$\begin{aligned} &1.27 \sum_{\substack{d_1,d_2<y\\ \gcd(d_1d_2,w)=1}} \sum_{\substack{s_1<y/d_1\\ s_2<y/d_2\\ \gcd(s_1s_2,w)=1}} \sum_{d_3|w} \sqrt{\frac{z}{d_1^2 s_1 d_2^2 s_2 d_3}} \\ &= 1.27\sqrt{z} \prod_{p|w}\left(1 + \frac{1}{\sqrt{p}}\right) \cdot \left(\sum_{\substack{d<y\\ \gcd(d,w)=1}} \sum_{\substack{s<y/d\\ \gcd(s,w)=1}} \frac{1}{d\sqrt{s}}\right)^2. \end{aligned}$$

We can bound the double sum simply by

$$\sum_{\substack{d<y\\ \gcd(d,w)=1}} \sum_{s<y/d} \frac{1}{\sqrt{s}d} \leq 2\sum_{d<y} \frac{\sqrt{y/d}}{d} \leq 2\sqrt{y}\cdot\zeta\left(\frac{3}{2}\right)\prod_{p|w}\left(1 - \frac{1}{p^{3/2}}\right).$$

Alternatively, if $w = 2$, we bound

$$\sum_{\substack{s<y/d\\ \gcd(s,w)=1}} \frac{1}{\sqrt{s}} = \sum_{\substack{s<y/d\\ s \text{ odd}}} \frac{1}{\sqrt{s}} \leq 1 + \frac{1}{2}\int_1^{y/d} \frac{1}{\sqrt{s}} ds = \sqrt{y/d}$$

and thus

$$\sum_{\substack{d<y\\ \gcd(d,w)=1}} \sum_{\substack{s<y/d\\ \gcd(s,w)=1}} \frac{1}{\sqrt{s}d} \leq \sum_{\substack{d<y\\ \gcd(d,2)=1}} \frac{\sqrt{y/d}}{d} \leq \sqrt{y}\left(1 - \frac{1}{2^{3/2}}\right)\zeta\left(\frac{3}{2}\right).$$

□

Applying Lemma 4.1 with $y = S/s$ and $z = x/Ws$, where $S = x/WU$, we obtain that (4.8) equals
(4.18)
$$\frac{6x}{\pi^2 W}\frac{w}{\sigma(w)} \sum_{\substack{s<S\\ \gcd(s,w)=1}} \frac{1}{s} \sum_{\substack{r_1<S/s\\ \gcd(r_1,r_2)=1\\ \gcd(r_1r_2,w)=1}} \sum_{r_2<S/s} \frac{\mu(r_1)\mu(r_2)}{\sigma(r_1)\sigma(r_2)}\left(1-\max\left(\frac{1}{2},\frac{r_1}{S/s},\frac{r_2}{S/s}\right)\right)$$
$$+O_{\leq}\left(5.04\zeta\left(\frac{3}{2}\right)^3 S\sqrt{\frac{x}{W}}\prod_{p|w}\left(1+\frac{1}{\sqrt{p}}\right)\left(1-\frac{1}{p^{3/2}}\right)^3\right),$$

with 5.04 replaced by 1.27 if $w = 2$. The main term in (4.18) can be written as

$$\frac{6x}{\pi^2 W}\frac{w}{\sigma(w)} \sum_{\substack{s\leq S\\ \gcd(s,w)=1}} \frac{1}{s}\int_{1/2}^{1} \sum_{\substack{r_1\leq \frac{uS}{s}\\ \gcd(r_1,r_2)=1\\ \gcd(r_1r_2,w)=1}} \sum_{r_2\leq\frac{uS}{s}} \frac{\mu(r_1)\mu(r_2)}{\sigma(r_1)\sigma(r_2)} du. \tag{4.19}$$

From now on, we will focus on the cases $w = 1$ and $w = 2$ for simplicity. (Higher values of w do not seem to be really profitable in the last analysis.)

4.1.2. *Explicit bounds for a sum with μ.* We must estimate the expression within parentheses in (4.19). It is not too hard to show that it tends to 0; the first part of the proof of Lemma 4.2 will reduce this to the fact that $\sum_n \mu(n)/n = 0$. Obtaining good bounds is a more delicate matter. For our purposes, we will need the expression to converge to 0 at least as fast as $1/(\log)^2$, with a good constant in front. For this task, the bound (2.6) on $\sum_{n\leq x}\mu(n)/n$ is enough.

Lemma 4.2. *Let*
$$g_w(x) := \sum_{\substack{r_1\leq x\\ \gcd(r_1,r_2)=1\\ \gcd(r_1r_2,w)=1}} \sum_{r_2\leq x} \frac{\mu(r_1)\mu(r_2)}{\sigma(r_1)\sigma(r_2)},$$

where $w = 1$ or $w = 2$. Then
$$|g_1(x)| \leq \begin{cases} 1/x & \text{if } 33\leq x\leq 10^6,\\ \frac{1}{x}(111.536+55.768\log x) & \text{if } 10^6\leq x<10^{10},\\ \frac{0.0044325}{(\log x)^2}+\frac{0.1079}{\sqrt{x}} & \text{if } x\geq 10^{10},\end{cases}$$

$$|g_2(x)| \leq \begin{cases} 2.1/x & \text{if } 1\leq x\leq 10^6,\\ \frac{1}{x}(1634.34+817.168\log x) & \text{if } 10^6\leq x<10^{10},\\ \frac{0.038128}{(\log x)^2}+\frac{0.2046}{\sqrt{x}}. & \text{if } x\geq 10^{10}.\end{cases}$$

Tbe proof involves what may be called a version of Rankin's trick, using Dirichlet series and the behavior of $\zeta(s)$ near $s = 1$. The statements for $x \leq 10^6$ are proven by direct computation.

Proof. Clearly

$$\begin{aligned}
g(x) &= \mathop{\sum_{r_1\le x}\sum_{r_2\le x}}_{\gcd(r_1r_2,w)=1} \left(\sum_{d|\gcd(r_1,r_2)} \mu(d)\right) \frac{\mu(r_1)\mu(r_2)}{\sigma(r_1)\sigma(r_2)} \\
&= \sum_{\substack{d\le x\\ \gcd(d,w)=1}} \mu(d) \mathop{\sum_{r_1\le x}\sum_{r_2\le x}}_{\substack{d|\gcd(r_1,r_2)\\ \gcd(r_1r_2,w)=1}} \frac{\mu(r_1)\mu(r_2)}{\sigma(r_1)\sigma(r_2)} \\
&= \sum_{\substack{d\le x\\ \gcd(d,w)=1}} \frac{\mu(d)}{(\sigma(d))^2} \sum_{\substack{u_1\le x/d\\ \gcd(u_1,dw)=1}} \sum_{\substack{u_2\le x/d\\ \gcd(u_2,dw)=1}} \frac{\mu(u_1)\mu(u_2)}{\sigma(u_1)\sigma(u_2)} \qquad (4.20)\\
&= \sum_{\substack{d\le x\\ \gcd(d,w)=1}} \frac{\mu(d)}{(\sigma(d))^2} \left(\sum_{\substack{r\le x/d\\ \gcd(r,dw)=1}} \frac{\mu(r)}{\sigma(r)}\right)^2 .
\end{aligned}$$

Moreover,

$$\begin{aligned}
\sum_{\substack{r\le x/d\\ \gcd(r,dw)=1}} \frac{\mu(r)}{\sigma(r)} &= \sum_{\substack{r\le x/d\\ \gcd(r,dw)=1}} \frac{\mu(r)}{r} \sum_{d'|r} \prod_{p|d'} \left(\frac{p}{p+1} - 1\right) \\
&= \sum_{\substack{d'\le x/d\\ \mu(d')^2=1\\ \gcd(d',dw)=1}} \left(\prod_{p|d'} \frac{-1}{p+1}\right) \sum_{\substack{r\le x/d\\ \gcd(r,dw)=1\\ d'|r}} \frac{\mu(r)}{r} \\
&= \sum_{\substack{d'\le x/d\\ \mu(d')^2=1\\ \gcd(d',dw)=1}} \frac{1}{d'\sigma(d')} \sum_{\substack{r\le x/dd'\\ \gcd(r,dd'w)=1}} \frac{\mu(r)}{r}
\end{aligned}$$

and

$$\sum_{\substack{r\le x/dd'\\ \gcd(r,dd'w)=1}} \frac{\mu(r)}{r} = \sum_{\substack{d''\le x/dd'\\ d''|(dd'w)^\infty}} \frac{1}{d''} \sum_{r\le x/dd'd''} \frac{\mu(r)}{r}.$$

Hence

(4.21)

$$|g(x)| \le \sum_{\substack{d\le x\\ \gcd(d,w)=1}} \frac{(\mu(d))^2}{(\sigma(d))^2} \left(\sum_{\substack{d'\le x/d\\ \mu(d')^2=1\\ \gcd(d',dw)=1}} \frac{1}{d'\sigma(d')} \sum_{\substack{d''\le x/dd'\\ d''|(dd'w)^\infty}} \frac{1}{d''} f(x/dd'd'')\right)^2 ,$$

where $f(t) = \left|\sum_{r\le t} \mu(r)/r\right|$.

We intend to bound the function $f(t)$ by a linear combination of terms of the form $t^{-\delta}$, $\delta \in [0, 1/2)$. Thus it makes sense now to estimate $F_w(s_1, s_2, x)$, defined

to be the quantity

$$\sum_{\substack{d\\ \gcd(d,w)=1}} \frac{(\mu(d))^2}{(\sigma(d))^2} \left(\sum_{\substack{d_1'\\ \gcd(d_1',dw)=1}} \frac{\mu(d_1')^2}{d_1'\sigma(d_1')} \sum_{d_1''|(dd_1'w)^\infty} \frac{1}{d_1''}\cdot (dd_1'd_1'')^{1-s_1}\right)$$
$$\left(\sum_{\substack{d_2'\\ \gcd(d_2',dw)=1}} \frac{\mu(d_2')^2}{d_2'\sigma(d_2')} \sum_{d_2''|(dd_2'w)^\infty} \frac{1}{d_2''}\cdot (dd_2'd_2'')^{1-s_2}\right).$$

for $s_1, s_2 \in [1/2, 1]$. This is equal to

$$\sum_{\substack{d\\ \gcd(d,w)=1}} \frac{\mu(d)^2}{d^{s_1+s_2}} \prod_{p|d} \frac{1}{(1+p^{-1})^2 (1-p^{-s_1}) \prod_{p|w} \frac{1}{(1-p^{-s_1})(1-p^{-s_2})} (1-p^{-s_2})}$$
$$\cdot \left(\sum_{\substack{d'\\ \gcd(d',dw)=1}} \frac{\mu(d')^2}{(d')^{s_1+1}} \prod_{p'|d'} \frac{1}{(1+p'^{-1})(1-p'^{-s_1})}\right)$$
$$\cdot \left(\sum_{\substack{d'\\ \gcd(d',dw)=1}} \frac{\mu(d')^2}{(d')^{s_2+1}} \prod_{p'|d'} \frac{1}{(1+p'^{-1})(1-p'^{-s_2})}\right),$$

which in turn can easily be seen to equal

$$\begin{aligned}(4.22)\qquad &\prod_{p\nmid w}\left(1+\frac{p^{-s_1}p^{-s_2}}{(1-p^{-s_1}+p^{-1})(1-p^{-s_2}+p^{-1})}\right)\prod_{p|w}\frac{1}{(1-p^{-s_1})(1-p^{-s_2})}\\ &\cdot \prod_{p\nmid w}\left(1+\frac{p^{-1}p^{-s_1}}{(1+p^{-1})(1-p^{-s_1})}\right)\cdot \prod_{p\nmid w}\left(1+\frac{p^{-1}p^{-s_2}}{(1+p^{-1})(1-p^{-s_2})}\right)\end{aligned}$$

Now, for any $0 < x \leq y \leq x^{1/2} < 1$,

$$(1+x-y)(1-xy)(1-xy^2)-(1+x)(1-y)(1-x^3) = (x-y)(y^2-x)(xy-x-1)x \leq 0,$$

and so

$$(4.23)\qquad 1+\frac{xy}{(1+x)(1-y)} = \frac{(1+x-y)(1-xy)(1-xy^2)}{(1+x)(1-y)(1-xy)(1-xy^2)} \leq \frac{(1-x^3)}{(1-xy)(1-xy^2)}.$$

For any $x \leq y_1, y_2 < 1$ with $y_1^2 \leq x$, $y_2^2 \leq x$,

$$(4.24)\qquad 1+\frac{y_1y_2}{(1-y_1+x)(1-y_2+x)} \leq \frac{(1-x^3)^2(1-x^4)}{(1-y_1y_2)(1-y_1y_2^2)(1-y_1^2y_2)}.$$

This can be checked as follows: multiplying by the denominators and changing variables to x, $s = y_1 + y_2$ and $r = y_1y_2$, we obtain an inequality where the left side, quadratic on s with positive leading coefficient, must be less than or equal to the right side, which is linear on s. The left side minus the right side can be maximal for given x, r only when s is maximal or minimal. This happens when $y_1 = y_2$ or when either $y_i = \sqrt{x}$ or $y_i = x$ for at least one of $i = 1, 2$. In each of

these cases, we have reduced (4.24) to an inequality in two variables that can be proven automatically[2] by a quantifier-elimination program; the author has used QEPCAD [HB11] to do this.

Hence $F_w(s_1, s_2, x)$ is at most

$$\begin{aligned}&\prod_{p\nmid w} \frac{(1-p^{-3})^2(1-p^{-4})}{(1-p^{-s_1-s_2})(1-p^{-2s_1-s_2})(1-p^{-s_1-2s_2})} \cdot \prod_{p|w} \frac{1}{(1-p^{-s_1})(1-p^{-s_2})}\\ &\cdot \prod_{p\nmid w} \frac{1-p^{-3}}{(1+p^{-s_1-1})(1+p^{-2s_1-1})} \prod_{p\nmid w} \frac{1-p^{-3}}{(1+p^{-s_2-1})(1+p^{-2s_2-1})}\\ &= C_{w,s_1,s_2} \cdot \frac{\zeta(s_1+1)\zeta(s_2+1)\zeta(2s_1+1)\zeta(2s_2+1)}{\zeta(3)^4\zeta(4)(\zeta(s_1+s_2)\zeta(2s_1+s_2)\zeta(s_1+2s_2))^{-1}},\end{aligned} \tag{4.25}$$

where

$$C_{w,s_1,s_2} = \begin{cases} 1 & \text{if } w = 1,\\ \frac{(1-2^{-s_1-2s_2})(1+2^{-s_1-1})(1+2^{-2s_1-1})(1+2^{-s_2-1})(1+2^{-2s_2-1})}{(1-2^{-s_1+s_2})^{-1}(1-2^{-2s_1-s_2})^{-1}(1-2^{-s_1})(1-2^{-s_2})(1-2^{-3})^4(1-2^{-4})} & \text{if } w = 2.\end{cases}$$

For $1 \leq t \leq x$, (2.6) and (2.9) imply

$$f(t) \leq \begin{cases} \sqrt{\frac{2}{t}} & \text{if } x \leq 10^{10}\\ \sqrt{\frac{2}{t}} + \frac{0.03}{\log x}\left(\frac{x}{t}\right)^{\frac{\log\log 10^{10}}{\log x - \log 10^{10}}} & \text{if } x > 10^{10},\end{cases} \tag{4.26}$$

where we are using the fact that $\log x$ is convex-down. Note that, again by convexity,

$$\frac{\log\log x - \log\log 10^{10}}{\log x - \log 10^{10}} < (\log t)'|_{t=\log 10^{10}} = \frac{1}{\log 10^{10}} = 0.0434294\ldots$$

Obviously, $\sqrt{2/t}$ in (4.26) can be replaced by $(2/t)^{1/2-\epsilon}$ for any $\epsilon \geq 0$.

By (4.21) and (4.26),

$$|g_w(x)| \leq \left(\frac{2}{x}\right)^{1-2\epsilon} F_w(1/2+\epsilon, 1/2+\epsilon, x)$$

for $x \leq 10^{10}$. We set $\epsilon = 1/\log x$ and obtain from (4.25) that

$$\begin{aligned}F(1/2+\epsilon, 1/2+\epsilon, x) &\leq C_{w,\frac{1}{2}+\epsilon,\frac{1}{2}+\epsilon} \frac{\zeta(1+2\epsilon)\zeta(3/2)^4\zeta(2)^2}{\zeta(3)^4\zeta(4)}\\ &\leq 55.768 \cdot C_{w,\frac{1}{2}+\epsilon,\frac{1}{2}+\epsilon} \cdot \left(1 + \frac{\log x}{2}\right),\end{aligned} \tag{4.27}$$

where we use the easy bound $\zeta(s) < 1 + 1/(s-1)$ obtained by

$$\sum n^s < 1 + \int_1^\infty t^s dt.$$

(For sharper bounds, see [BR02].) Now

$$C_{2,\frac{1}{2}+\epsilon,\frac{1}{2}+\epsilon} \leq \frac{(1-2^{-3/2-\epsilon})^2(1+2^{-3/2})^2(1+2^{-2})^2(1-2^{-1-2\epsilon})}{(1-2^{-1/2})^2(1-2^{-3})^4(1-2^{-4})} \leq 14.652983,$$

[2]In practice, the case $y_i = \sqrt{x}$ leads to a polynomial of high degree, and quantifier elimination increases sharply in complexity as the degree increases; a stronger inequality of lower degree (with $(1-3x^3)$ instead of $(1-x^3)^2(1-x^4)$) was given to QEPCAD to prove in this case.

whereas $C_{1,\frac{1}{2}+\epsilon,\frac{1}{2}+\epsilon} = 1$. (We are assuming $x \geq 10^6$, and so $\epsilon \leq 1/(\log 10^6)$.) Hence

$$|g_w(x)| \leq \begin{cases} \frac{1}{x}(111.536 + 55.768 \log x) & \text{if } w = 1, \\ \frac{1}{x}(1634.34 + 817.168 \log x) & \text{if } w = 2. \end{cases}$$

for $10^6 \leq x < 10^{10}$.

For general x, we must use the second bound in (4.26). Define $c = 1/(\log 10^{10})$. We see that, if $x > 10^{10}$,

$$\begin{aligned} |g_w(x)| \leq &\frac{0.03^2}{(\log x)^2} F_1(1-c, 1-c) \cdot C_{w,1-c,1-c} \\ &+ 2 \cdot \frac{\sqrt{2}}{\sqrt{x}} \frac{0.03}{\log x} F(1-c, 1/2) \cdot C_{w,1-c,1/2} \\ &+ \frac{1}{x}(111.536 + 55.768 \log x) \cdot C_{w,\frac{1}{2}+\epsilon,\frac{1}{2}+\epsilon}. \end{aligned}$$

For $w = 1$, this gives

$$\begin{aligned} |g_1(x)| &\leq \frac{0.0044325}{(\log x)^2} + \frac{2.1626}{\sqrt{x}\log x} + \frac{1}{x}(111.536 + 55.768 \log x) \\ &\leq \frac{0.0044325}{(\log x)^2} + \frac{0.1079}{\sqrt{x}}; \end{aligned}$$

for $w = 2$, we obtain

$$\begin{aligned} |g_2(x)| &\leq \frac{0.038128}{(\log x)^2} + \frac{25.607}{\sqrt{x}\log x} + \frac{1}{x}(1634.34 + 817.168 \log x) \\ &\leq \frac{0.038128}{(\log x)^2} + \frac{0.2046}{\sqrt{x}}. \end{aligned}$$

□

4.1.3. *Estimating the triple sum.* We will now be able to bound the triple sum in (4.19), viz.,

$$\sum_{\substack{s \leq S \\ \gcd(s,w)=1}} \frac{1}{s} \int_{1/2}^{1} g_w(uS/s) du, \tag{4.28}$$

where g_w is as in Lemma 4.2.

As we will soon see, Lemma 4.2 that (4.28) is bounded by a constant (essentially because the integral $\int_0^{1/2} 1/t(\log t)^2$ converges). We must give as good a constant as we can, since it will affect the largest term in the final result.

Clearly $g_w(R) = g_w(\lfloor R \rfloor)$. The contribution of each $g_w(m)$, $1 \leq m \leq S$, to (4.28) is exactly $g_w(m)$ times

(4.29)

$$\begin{aligned} &\sum_{\substack{\frac{S}{m+1} < s \leq \frac{S}{m} \\ \gcd(s,w)=1}} \frac{1}{s} \int_{ms/S}^{1} du + \sum_{\substack{\frac{S}{2m} < s \leq \frac{S}{m+1} \\ \gcd(s,w)=1}} \frac{1}{s} \int_{ms/S}^{(m+1)s/S} du + \sum_{\substack{\frac{S}{2(m+1)} < s \leq \frac{S}{2m} \\ \gcd(s,w)=1}} \frac{1}{s} \int_{1/2}^{(m+1)s/S} du \\ &= \sum_{\substack{\frac{S}{m+1} < s \leq \frac{S}{m} \\ \gcd(s,w)=1}} \left(\frac{1}{s} - \frac{m}{S}\right) + \sum_{\substack{\frac{S}{2m} < s \leq \frac{S}{m+1} \\ \gcd(s,w)=1}} \frac{1}{S} + \sum_{\substack{\frac{S}{2(m+1)} < s \leq \frac{S}{2m} \\ \gcd(s,w)=1}} \left(\frac{m+1}{S} - \frac{1}{2s}\right). \end{aligned}$$

Write $f(t) = 1/S$ for $S/2m < t \leq S/(m+1)$, $f(t) = 0$ for $t > S/m$ or $t < S/2(m+1)$, $f(t) = 1/t - m/S$ for $S/(m+1) < t \leq S/m$ and $f(t) = (m+1)/S - 1/2t$ for $S/2(m+1) < t \leq S/2m$; then (4.29) equals $\sum_{n:\gcd(n,w)=1} f(n)$. By Euler-Maclaurin (second order),

(4.30)

$$\begin{aligned}\sum_n f(n) &= \int_{-\infty}^{\infty} f(x) - \frac{1}{2}B_2(\{x\})f''(x)dx = \int_{-\infty}^{\infty} f(x) + O_\leq\left(\frac{1}{12}|f''(x)|\right)dx\\ &= \int_{-\infty}^{\infty} f(x)dx + \frac{1}{6}\cdot O_\leq\left(\left|f'\left(\frac{3}{2m}\right)\right| + \left|f'\left(\frac{s}{m+1}\right)\right|\right)\\ &= \frac{1}{2}\log\left(1+\frac{1}{m}\right) + \frac{1}{6}\cdot O_\leq\left(\left(\frac{2m}{s}\right)^2 + \left(\frac{m+1}{s}\right)^2\right).\end{aligned}$$

Similarly,

$$\begin{aligned}\sum_{n \text{ odd}} f(n) &= \int_{-\infty}^{\infty} f(2x+1) - \frac{1}{2}B_2(\{x\})\frac{d^2 f(2x+1)}{dx^2}dx\\ &= \frac{1}{2}\int_{-\infty}^{\infty} f(x)dx - 2\int_{-\infty}^{\infty}\frac{1}{2}B_2\left(\left\{\frac{x-1}{2}\right\}\right)f''(x)dx\\ &= \frac{1}{2}\int_{-\infty}^{\infty} f(x)dx + \frac{1}{6}\int_{-\infty}^{\infty} O_\leq\left(|f''(x)|\right)dx\\ &= \frac{1}{4}\log\left(1+\frac{1}{m}\right) + \frac{1}{3}\cdot O_\leq\left(\left(\frac{2m}{s}\right)^2 + \left(\frac{m+1}{s}\right)^2\right).\end{aligned}$$

We use these expressions for $m \leq C_0$, where $C_0 \geq 33$ is a constant to be computed later; they will give us the main term. For $m > C_0$, we use the bounds on $|g(m)|$ that Lemma 4.2 gives us.

(Starting now and for the rest of the paper, we will focus on the cases $w = 1$, $w = 2$ when giving explicit computational estimates. All of our procedures would allow higher values of w as well, but, as will become clear much later, the gains from higher values of w are offset by losses and complications elsewhere.)

Let us estimate (4.28). Let

$$c_{w,0} = \begin{cases}1/6 & \text{if } w=1,\\ 1/3 & \text{if } w=2,\end{cases}\qquad c_{w,1} = \begin{cases}1 & \text{if } w=1,\\ 2.5 & \text{if } w=2,\end{cases}$$

$$c_{w,2} = \begin{cases}55.768\ldots & \text{if } w=1,\\ 817.168\ldots & \text{if } w=2,\end{cases}\qquad c_{w,3} = \begin{cases}111.536\ldots & \text{if } w=1,\\ 1634.34\ldots & \text{if } w=2,\end{cases}$$

$$c_{w,4} = \begin{cases}0.0044325\ldots & \text{if } w=1,\\ 0.038128\ldots & \text{if } w=2,\end{cases}\qquad c_{w,5} = \begin{cases}0.1079\ldots & \text{if } w=1,\\ 0.2046\ldots & \text{if } w=2.\end{cases}$$

Then (4.28) equals

$$\begin{aligned}&\sum_{m\leq C_0} g_w(m)\cdot\left(\frac{\phi(w)}{2w}\log\left(1+\frac{1}{m}\right)+O_\leq\left(c_{w,0}\frac{5m^2+2m+1}{S^2}\right)\right)\\&\quad+\sum_{S/10^6\leq s<S/C_0}\frac{1}{s}\int_{1/2}^1 O_\leq\left(\frac{c_{w,1}}{uS/s}\right)du\\&\quad+\sum_{S/10^{10}\leq s<S/10^6}\frac{1}{s}\int_{1/2}^1 O_\leq\left(\frac{c_{w,2}\log(uS/s)+c_{w,3}}{uS/s}\right)du\\&\quad+\sum_{s<S/10^{10}}\frac{1}{s}\int_{1/2}^1 O_\leq\left(\frac{c_{w,4}}{(\log uS/s)^2}+\frac{c_{w,5}}{\sqrt{uS/s}}\right)du,\end{aligned}$$

which is

$$\begin{aligned}&\sum_{m\leq C_0} g_w(m)\cdot\frac{\phi(w)}{2w}\log\left(1+\frac{1}{m}\right)+\sum_{m\leq C_0}|g(m)|\cdot O_\leq\left(c_{w,0}\frac{5m^2+2m+1}{S^2}\right)\\&\quad+O_\leq\left(c_{w,1}\frac{\log 2}{C_0}+\frac{\log 2}{10^6}\left(c_{w,3}+c_{w,2}(1+\log 10^6)\right)+\frac{2-\sqrt{2}}{10^{10/2}}c_{w,5}\right)\\&\quad+O_\leq\left(\sum_{s<S/10^{10}}\frac{c_{w,4}/2}{s(\log S/2s)^2}\right)\end{aligned}$$

for $S\geq(C_0+1)$. Note that $\sum_{s<S/10^{10}}\frac{1}{s(\log S/2s)^2}=\int_0^{2/10^{10}}\frac{1}{t(\log t)^2}dt$.

Now

$$\frac{c_{w,4}}{2}\int_0^{2/10^{10}}\frac{1}{t(\log t)^2}dt=\frac{c_{w,4}/2}{\log(10^{10}/2)}=\begin{cases}0.00009923\ldots & \text{if } w=1\\ 0.000853636\ldots & \text{if } w=2.\end{cases}$$

and

$$\frac{\log 2}{10^6}\left(c_{w,3}+c_{w,2}(1+\log 10^6)\right)+\frac{2-\sqrt{2}}{10^5}c_{w,5}=\begin{cases}0.0006506\ldots & \text{if } w=1\\ 0.009525\ldots & \text{if } w=2.\end{cases}$$

For $C_0=10000$,

$$\frac{\phi(w)}{w}\frac{1}{2}\sum_{m\leq C_0} g_w(m)\cdot\log\left(1+\frac{1}{m}\right)=\begin{cases}0.362482\ldots & \text{if } w=1,\\ 0.360576\ldots & \text{if } w=2,\end{cases}$$

$$c_{w,0}\sum_{m\leq C_0}|g_w(m)|(5m^2+2m+1)\leq\begin{cases}6204066.5\ldots & \text{if } w=1,\\ 15911340.1\ldots & \text{if } w=2,\end{cases}$$

and

$$c_{w,1}\cdot(\log 2)/C_0=\begin{cases}0.00006931\ldots & \text{if } w=1,\\ 0.00017328\ldots & \text{if } w=2.\end{cases}$$

Thus, for $S\geq 100000$,

$$\sum_{s\leq S}\frac{1}{s}\int_{1/2}^1 g_w(uS/s)du\leq\begin{cases}0.36393 & \text{if } w=1,\\ 0.37273 & \text{if } w=2.\end{cases}\tag{4.31}$$

For $S<100000$, we proceed as above, but using the exact expression (4.29) instead of (4.30). Note (4.29) is of the form $f_{s,m,1}(S)+f_{s,m,2}(S)/S$, where both

$f_{s,m,1}(S)$ and $f_{s,m,2}(S)$ depend only on $\lfloor S \rfloor$ (and on s and m). Summing over $m \leq S$, we obtain a bound of the form

$$\sum_{s \leq S} \frac{1}{s} \int_{1/2}^{1} g_w(uS/s) du \leq G_w(S)$$

with

$$G_w(S) = K_{w,1}(\lfloor S \rfloor) + K_{w,2}(\lfloor S \rfloor)/S,$$

where $K_{w,1}(n)$ and $K_{w,2}(n)$ can be computed explicitly for each integer n. (For example, $G_w(S) = 1 - 1/S$ for $1 \leq S < 2$ and $G_w(S) = 0$ for $S < 1$.)

It is easy to check numerically that this implies that (4.31) holds not just for $S \geq 100000$ but also for $40 \leq S < 100000$ (if $w = 1$) or $16 \leq S < 100000$ (if $w = 2$). Using the fact that $G_w(S)$ is non-negative, we can compare $\int_1^T G_w(S) dS/S$ with $\log(T + 1/N)$ for each $T \in [2, 40] \cap \frac{1}{N}\mathbb{Z}$ (N a large integer) to show, again numerically, that

$$\int_1^T G_w(S) \frac{dS}{S} \leq \begin{cases} 0.3698 \log T & \text{if } w = 1, \\ 0.37273 \log T & \text{if } w = 2. \end{cases} \tag{4.32}$$

(We use $N = 100000$ for $w = 1$; already $N = 10$ gives us the answer above for $w = 2$. Indeed, computations suggest the better bound 0.358 instead of 0.37273; we are committed to using 0.37273 because of (4.31).)

Multiplying by $6w/\pi^2\sigma(w)$, we conclude that

$$S_1(U, W) = \frac{x}{W} \cdot H_1\left(\frac{x}{WU}\right) + O_{\leq}\left(5.08\zeta(3/2)^3 \frac{x^{3/2}}{W^{3/2}U}\right) \tag{4.33}$$

if $w = 1$,

$$S_1(U, W) = \frac{x}{W} \cdot H_2\left(\frac{x}{WU}\right) + O_{\leq}\left(1.27\zeta(3/2)^3 \frac{x^{3/2}}{W^{3/2}U}\right) \tag{4.34}$$

if $w = 2$, where
(4.35)

$$H_1(S) = \begin{cases} \frac{6}{\pi^2} G_1(S) & \text{if } 1 \leq S < 40, \\ 0.22125 & \text{if } S \geq 40, \end{cases} \qquad H_2(s) = \begin{cases} \frac{4}{\pi^2} G_2(S) & \text{if } 1 \leq S < 16, \\ 0.15107 & \text{if } S \geq 16. \end{cases}$$

Hence (by (4.32))

$$\int_1^T H_w(S) \frac{dS}{S} \leq \begin{cases} 0.22482 \log T & \text{if } w = 1, \\ 0.15107 \log T & \text{if } w = 2; \end{cases} \tag{4.36}$$

moreover, $H_1(S) \leq 3/\pi^2$, $H_2(S) \leq 2/\pi^2$ for all S.

* * *

Note. There is another way to obtain cancellation on μ, applicable when $(x/W) > Uq$ (as is unfortunately never the case in our main application). For this alternative to be taken, one must either apply Cauchy-Schwarz on n rather than m (resulting in exponential sums over m) or lump together all m near each other and in the same congruence class modulo q before applying Cauchy-Schwarz on m (one can indeed do this if δ is small). We could then write

$$\sum_{\substack{m \sim W \\ m \equiv r \bmod q}} \sum_{\substack{d|m \\ d > U}} \mu(d) = - \sum_{\substack{m \sim W \\ m \equiv r \bmod q}} \sum_{\substack{d|m \\ d \leq U}} \mu(d) = - \sum_{d \leq U} \mu(d)(W/qd + O(1))$$

and obtain cancellation on d. If $Uq \geq (x/W)$, however, the error term dominates.

4.2. **The sum S_2: the large sieve, primes and tails.** We must now bound

$$S_2(U',W',W) = \sum_{U'<m\leq \frac{x}{W}} \left|\sum_{W'<p\leq W} (\log p) e(\alpha m p)\right|^2 . \tag{4.37}$$

for $U' = \max(U, x/2W)$, $W' = \max(S, W/2)$.

From a modern perspective, this is clearly a case for a large sieve. It is also clear that we ought to try to apply a large sieve for sequences of prime support. What is subtler here is how to do things well for very large q (i.e., x/q small). This is in some sense a dual problem to that of q small, but it poses additional complications; for example, it is not obvious how to take advantage of prime support for very large q.

As in type I, we avoid this entire issue by forbidding q large and then taking advantage of the error term δ/x in the approximation $\alpha = \frac{a}{q} + \frac{\delta}{x}$. This is one of the main innovations here. Note this alternative method will allow us to take advantage of prime support.

A key situation to study is that of frequencies α_i clustering around given rationals a/q while nevertheless keeping at a certain small distance from each other.

Lemma 4.3. *Let $q \geq 1$. Let $\alpha_1, \alpha_2, \dots, \alpha_k \in \mathbb{R}/\mathbb{Z}$ be of the form $\alpha_i = a_i/q + \upsilon_i$, $0 \leq a_i < q$, where the elements $\upsilon_i \in \mathbb{R}$ all lie in an interval of length $\upsilon > 0$, and where $a_i = a_j$ implies $|\upsilon_i - \upsilon_j| > \nu > 0$. Assume $\nu + \upsilon \leq 1/q$. Then, for any $W, W' \geq 1$, $W' \geq W/2$,*

$$\begin{aligned}\sum_{i=1}^{k} \left|\sum_{W'<p\leq W} (\log p) e(\alpha_i p)\right|^2 &\leq \min\left(1, \frac{2q}{\phi(q)} \frac{1}{\log\left((q(\nu+\upsilon))^{-1}\right)}\right) \\ &\quad \cdot \left(W - W' + \nu^{-1}\right) \sum_{W'<p\leq W} (\log p)^2.\end{aligned} \tag{4.38}$$

Proof. For any distinct i, j, the angles α_i, α_j are separated by at least ν (if $a_i = a_j$) or at least $1/q - |\upsilon_i - \upsilon_j| \geq 1/q - \upsilon \geq \nu$ (if $a_i \neq a_j$). Hence we can apply the large sieve (in the optimal $N + \delta^{-1} - 1$ form due to Selberg [Sel91] and Montgomery-Vaughan [MV74]) and obtain the bound in (4.38) with 1 instead of $\min(1, \dots)$ immediately.

We can also apply Montgomery's inequality ([Mon68], [Hux72]; see the expositions in [Mon71, pp. 27–29] and [IK04, §7.4]). This gives us that the left side of (4.38) is at most

(4.39)
$$\left(\sum_{\substack{r\leq R \\ \gcd(r,q)=1}} \frac{(\mu(r))^2}{\phi(r)}\right)^{-1} \sum_{\substack{r\leq R \\ \gcd(r,q)=1}} \sum_{\substack{a' \bmod r \\ \gcd(a',r)=1}} \sum_{i=1}^{k} \left|\sum_{W'<p\leq W} (\log p) e((\alpha_i + a'/r)p)\right|^2$$

If we add all possible fractions of the form a'/r, $r \leq R$, $\gcd(r,q) = 1$, to the fractions a_i/q, we obtain fractions that are separated by at least $1/qR^2$. If $\nu + \upsilon \geq 1/qR^2$, then the resulting angles $\alpha_i + a'/r$ are still separated by at least ν. Thus

we can apply the large sieve to (4.39); setting $R = 1/\sqrt{(\nu+\upsilon)q}$, we see that we gain a factor of

(4.40)
$$\sum_{\substack{r\leq R\\ \gcd(r,q)=1}} \frac{(\mu(r))^2}{\phi(r)} \geq \frac{\phi(q)}{q}\sum_{r\leq R}\frac{(\mu(r))^2}{\phi(r)} \geq \frac{\phi(q)}{q}\sum_{d\leq R}\frac{1}{d} \geq \frac{\phi(q)}{2q}\log\left((q(\nu+\upsilon))^{-1}\right),$$

since $\sum_{d\leq R} 1/d \geq \log(R)$ for all $R\geq 1$ (integer or not). □

Let us first give a bound on sums of the type of $S_2(U,V,W)$ using prime support but not the error terms (or Lemma 4.3).

Lemma 4.4. *Let $W\geq 1$, $W'\geq W/2$. Let $\alpha = a/q + O_\leq(1/qQ)$, $q\leq Q$. Then*

$$\begin{aligned}\sum_{A_0<m\leq A_1}\left|\sum_{W'<p\leq W}(\log p)e(\alpha m p)\right|^2 \\ \leq \left\lceil\frac{A_1-A_0}{\min(q,\lceil Q/2\rceil)}\right\rceil\cdot(W-W'+2q)\sum_{W'<p\leq W}(\log p)^2.\end{aligned}\tag{4.41}$$

If $q<W/2$ and $Q\geq 3.5W$, the following bound also holds:

$$\begin{aligned}\sum_{A_0<m\leq A_1}\left|\sum_{W'<p\leq W}(\log p)e(\alpha m p)\right|^2 \\ \leq \left\lceil\frac{A_1-A_0}{q}\right\rceil\cdot\frac{q}{\phi(q)}\frac{W}{\log(W/2q)}\cdot\sum_{W'<p\leq W}(\log p)^2.\end{aligned}\tag{4.42}$$

If $A_1-A_0\leq|\delta q|$ and $q\leq\rho Q$, $\delta,\rho\in[0,1]$, the following bound also holds:

$$\begin{aligned}\sum_{A_0<m\leq A_1}\left|\sum_{W'<p\leq W}(\log p)e(\alpha m p)\right|^2 \\ \leq (W-W'+q/(1-\delta\rho))\sum_{W'<p\leq W}(\log p)^2.\end{aligned}\tag{4.43}$$

The inequality (4.42) can be stronger than (4.42) only when $q<W/7.2638\ldots$ (if q is odd) or $q<W/92.514\ldots$ (if q is even).

Proof. Let $k=\min(q,\lceil Q/2\rceil)\geq\lceil q/2\rceil$. We split $(A_0,A_1]$ into $\lceil(A_1-A_0)/k\rceil$ blocks of at most k consecutive integers $m_0+1, m_0+2,\ldots$. For m, m' in such a block, αm and $\alpha m'$ are separated by a distance of at least

$$|\{(a/q)(m-m')\}| - O_\leq(k/qQ) = 1/q - O_\leq(1/2q) \geq 1/2q.$$

By the large sieve

$$\sum_{a=1}^{q}\left|\sum_{W'<p\leq W}(\log p)e(\alpha(m_0+a)p)\right|^2 \leq ((W-W')+2q)\sum_{W'<p\leq W}(\log p)^2.\tag{4.44}$$

We obtain (4.41) by summing over all $\lceil(A_1-A_0)/k\rceil$ blocks.

If $A_1-A_0\leq|\delta q|$ and $q\leq\rho Q$, $\delta,\rho\in[0,1]$, we obtain (4.43) simply by applying the large sieve without splitting the interval $A_0<m\leq A_1$.

Let us now prove (4.42). We will use Montgomery's inequality, followed by Montgomery and Vaughan's large sieve with weights. An angle $a/q + a_1'/r_1$ is separated from other angles $a'/q + a_2'/r_2$ ($r_1, r_2 \leq R$, $\gcd(a_i, r_i) = 1$) by at least $1/qr_1R$, rather than just $1/qR^2$. We will choose R so that $qR^2 < Q$; this implies $1/Q < 1/qR^2 \leq 1/qr_1R$.

By Montgomery's inequality [IK04, Lemma 7.15], applied (for each $1 \leq a \leq q$) to $S(\alpha) = \sum_n a_n e(\alpha n)$ with $a_n = \log(n) e(\alpha(m_0 + a)n)$ if n is prime and $a_n = 0$ otherwise,

$$\begin{aligned} &\frac{1}{\phi(r)} \left| \sum_{W' < p \leq W} (\log p) e(\alpha(m_0 + a)p) \right|^2 \\ &\leq \sum_{\substack{a' \bmod r \\ \gcd(a', r) = 1}} \left| \sum_{W' < p \leq W} (\log p) e\left(\left(\alpha\left(m_0 + a\right) + \frac{a'}{r} \right) p \right) \right|^2 . \end{aligned} \tag{4.45}$$

for each square-free $r \leq W'$. We multiply both sides of (4.45) by $(W/2 + (3/2)(1/qrR - 1/Q)^{-1})^{-1}$ and sum over all $a = 0, 1, \ldots, q-1$ and all square-free $r \leq R$ coprime to q; we will later make sure that $R \leq W'$. We obtain that

$$\sum_{\substack{r \leq R \\ \gcd(r,q)=1}} \left(\frac{W}{2} + \frac{3}{2} \left(\frac{1}{qrR} - \frac{1}{Q} \right)^{-1} \right)^{-1} \frac{\mu(r)^2}{\phi(r)} \left| \sum_{W' < p \leq W} (\log p) e(\alpha(m_0 + a)p) \right|^2$$

is at most

$$\begin{aligned} &\sum_{\substack{r \leq R \\ \gcd(r,q)=1 \\ r \text{ sq-free}}} \left(\frac{W}{2} + \frac{3}{2} \left(\frac{1}{qrR} - \frac{1}{Q} \right)^{-1} \right)^{-1} \\ &\sum_{a=1}^{q} \sum_{\substack{a' \bmod r \\ \gcd(a', r) = 1}} \left| \sum_{W' < p \leq W} (\log p) e\left(\left(\alpha\left(m_0 + a\right) + \frac{a'}{r} \right) p \right) \right|^2 \end{aligned} \tag{4.46}$$

We now apply the large sieve with weights [MV73, (1.6)], recalling that each angle $\alpha(m_0 + a) + a'/r$ is separated from the others by at least $1/rR - 1/Q$; we obtain that (4.46) is at most $\sum_{W' < p \leq W} (\log p)^2$. (We have followed a procedure analogous to that used in [MV73] to prove the Brun-Titchmarsh theorem.)

Assume first that $q \leq W/13.5$. Set

$$R = \left(\sigma \frac{W}{q} \right)^{1/2}, \tag{4.47}$$

where $\sigma = 1/2e^{2 \cdot 0.25068} = 0.30285\ldots$. It is clear that $qR^2 < Q$, $q < W'$ and $R \geq 2$. Moreover, for $r \leq R$,

$$\frac{1}{Q} \leq \frac{1}{3.5W} \leq \frac{\sigma}{3.5} \frac{1}{\sigma W} = \frac{\sigma}{3.5} \frac{1}{qR^2} \leq \frac{\sigma/3.5}{qrR}.$$

Hence

$$\begin{aligned}\frac{W}{2} + \frac{3}{2}\left(\frac{1}{qrR} - \frac{1}{Q}\right)^{-1} &\leq \frac{W}{2} + \frac{3}{2}\frac{qrR}{1-\sigma/3.5} = \frac{W}{2} + \frac{3r}{2\left(1-\frac{\sigma}{3.5}\right)R}\cdot 2\sigma\frac{W}{2}\\ &= \frac{W}{2}\left(1+\frac{3\sigma}{1-\sigma/3.5}\frac{rW}{R}\right) < \frac{W}{2}\left(1+\frac{rW}{R}\right)\end{aligned}$$

and so

$$\begin{aligned}&\sum_{\substack{r\leq R\\ \gcd(r,q)=1}} \left(\frac{W}{2} + \frac{3}{2}\left(\frac{1}{qrR} - \frac{1}{Q}\right)^{-1}\right)^{-1}\frac{\mu(r)^2}{\phi(r)}\\ &\qquad \geq \frac{2}{W}\sum_{\substack{r\leq R\\ \gcd(r,q)=1}} (1+rR^{-1})^{-1}\frac{\mu(r)^2}{\phi(r)} \geq \frac{2}{W}\frac{\phi(q)}{q}\sum_{r\leq R}(1+rR^{-1})^{-1}\frac{\mu(r)^2}{\phi(r)}.\end{aligned}$$

For $R \geq 2$,

$$\sum_{r\leq R}(1+rR^{-1})^{-1}\frac{\mu(r)^2}{\phi(r)} > \log R + 0.25068;$$

this is true for $R \geq 100$ by [MV73, Lemma 8] and easily verifiable numerically for $2 \leq R < 100$. (It suffices to verify this for R integer with $r < R$ instead of $r \leq R$, as that is the worst case.)

Now

$$\log R = \frac{1}{2}\left(\log\frac{W}{2q} + \log 2\sigma\right) = \frac{1}{2}\log\frac{W}{2q} - 0.25068.$$

Hence

$$\sum_{r\leq R}(1+rR^{-1})^{-1}\frac{\mu(r)^2}{\phi(r)} > \frac{1}{2}\log\frac{W}{2q}$$

and the statement follows.

Now consider the case $q > W/13.5$. If q is even, then, in this range, inequality (4.41) is always better than (4.42), and so we are done. Assume, then, that $W/13.5 < q \leq W/2$ and q is odd. We set $R = 2$; clearly $qR^2 < W \leq Q$ and $q < W/2 \leq W'$, and so this choice of R is valid. It remains to check that

$$\frac{1}{\frac{W}{2}+\frac{3}{2}\left(\frac{1}{2q}-\frac{1}{Q}\right)^{-1}} + \frac{1}{\frac{W}{2}+\frac{3}{2}\left(\frac{1}{4q}-\frac{1}{Q}\right)^{-1}} \geq \frac{1}{W}\log\frac{W}{2q}.$$

This follows because

$$\frac{1}{\frac{1}{2}+\frac{3}{2}\left(\frac{t}{2}-\frac{1}{3.5}\right)^{-1}} + \frac{1}{\frac{1}{2}+\frac{3}{2}\left(\frac{t}{4}-\frac{1}{3.5}\right)^{-1}} \geq \log\frac{t}{2}$$

for all $2 \leq t \leq 13.5$.

□

The idea now (for large δ) is that, if δ is not negligible, then, as m increases, αm loops around the circle $\mathbb{R}/\mathbb{Z}$ roughly repeats itself every q steps – but with a slight displacement. This displacement gives rise to a configuration to which Lemma 4.3 is applicable.

Proposition 4.5. *Let $x \geq W \geq 1$, $W' \geq W/2$, $U' \geq x/2W$. Let $Q \geq 3.5W$. Let $\alpha = a/q + \delta/x$, $\gcd(a,q) = 1$, $|\delta/x| \leq 1/qQ$, $q \leq Q$. Let $S_2(U', W', W)$ be as in (4.37).*

For $q \leq \rho Q$, where $\rho \in [0,1]$,

$$S_2(U', W', W) \leq \left(\max(1, 2\rho)\left(\frac{x}{4q} + \frac{x}{W}\right) + \frac{W}{2} + 2q\right) \cdot \sum_{W' < p \leq W} (\log p)^2 \tag{4.48}$$

If $q < W/2$,

$$S_2(U', W', W) \leq \left(\frac{x}{2\phi(q)} \frac{1}{\log(W/2q)} + \frac{q}{\phi(q)} \frac{W}{\log(W/2q)}\right) \cdot \sum_{W' < p \leq W} (\log p)^2. \tag{4.49}$$

If $W > x/2q$, the following bound also holds:

$$S_2(U', W', W) \leq \left(\frac{W}{2} + \frac{q}{1 - x/2Wq}\right) \sum_{W' < p \leq W} (\log p)^2. \tag{4.50}$$

If $q \leq \rho Q$, where $\rho \in [0,1)$, and $\delta \neq 0$,

$$\begin{aligned} S_2(U', W', W) \leq \min\left(1, \frac{2q/\phi(q)}{\log\left(\frac{x}{|\delta q|}\left(q + \frac{x}{2W}\right)^{-1}\right)}\right) \\ \cdot \left(\frac{x}{|\delta q|} + \frac{W}{2} + \frac{x}{4(1-\rho)Q} + \frac{x}{2(1-\rho)W}\right) \sum_{W' < p \leq W} (\log p)^2. \end{aligned} \tag{4.51}$$

The trivial bound would be in the order of

$$S_2(U', W', W) = (x/2 \log x) \sum_{W' < p \leq W} (\log p)^2.$$

In practice, (4.50) gets applied when $W \geq x/q$.

Proof. Let us first prove statements (4.49) and (4.48), which do not involve δ. Assume first $q \leq W/2$. Then, by (4.42) with $A_0 = U'$, $A_1 = x/W$,

$$S_2(U', W', W) \leq \left(\frac{x/W - U'}{q} + 1\right) \frac{q}{\phi(q)} \frac{W}{\log(W/2q)} \sum_{W' < p \leq W} (\log p)^2.$$

Clearly $(x/W - U')W \leq (x/2W) \cdot W = x/2$. Thus (4.49) holds.

Assume now that $q \leq \rho Q$. Apply (4.41) with $A_0 = U'$, $A_1 = x/W$. Then

$$S_2(U', W', W) \leq \left(\frac{x/W - U'}{q \cdot \min(1, (2\rho)^{-1})} + 1\right)(W - W' + 2q) \sum_{W' < p \leq W} (\log p)^2.$$

Now

$$\begin{aligned} &\left(\frac{x/W - U'}{q \cdot \min(1, (2\rho)^{-1})} + 1\right) \cdot (W - W' + 2q) \\ &\leq \left(\frac{x}{W} - U'\right) \frac{W - W'}{q \min(1, (2\rho)^{-1})} + 2\max(1, 2\rho)\left(\frac{x}{W} - U'\right) + W/2 + 2q \\ &\leq \frac{x/4}{q \min(1, (2\rho)^{-1})} + \max(1, 2\rho)\frac{x}{W} + W/2 + 2q. \end{aligned}$$

This implies (4.48).

If $W > x/2q$, apply (4.43) with $\delta = x/2Wq$, $\rho = 1$. This yields (4.50).

Lastly, assume $\delta \neq 0$ and $q \leq \rho Q$. Let $Q' = x/|\delta q|$; then $Q' \geq Q$. For any m_1, m_2 with $|m_1 - m_2| \leq Q' - q$,

$$|(m_1 - m_2) \cdot \delta/x + q\delta/x| \leq Q'\delta/x \geq \frac{1}{q}.$$

We split the range $U' < m \leq x/W$ into $\lceil (x/W - U)/(Q' - q) \rceil$ intervals of length $\leq Q' - q$; in each interval, the conditions of Lemma 4.3 are fulfilled with $\upsilon = (x/2W) \cdot \delta/x$ and $\nu = |\delta q|/x$. We obtain a total upper bound of

$$S_2(U', W', W) \leq \min\left(1, \frac{2q}{\phi(q)} \frac{1}{\log\left((q(\nu+\upsilon))^{-1}\right)}\right) \cdot \left(1 + \frac{x/W - U}{Q' - q}\right)\left(W - W' + \nu^{-1}\right) \sum_{W' < p \leq W} (\log p)^2.$$

Here $W - W' + \nu^{-1} = W - W' + x/q\delta \leq W/2 + x/q\delta$ and

$$(q(\nu+\upsilon))^{-1} = \left(q\frac{\delta}{x}\right)^{-1}\left(q + \frac{x}{2W}\right)^{-1}.$$

Moreover,

$$\begin{aligned}\left(\frac{W}{2} + \frac{x}{q\delta}\right)\left(1 + \frac{x/W - U}{Q' - q}\right) &\leq \left(\frac{W}{2} + Q'\right)\left(1 + \frac{x/2W}{(1-\rho)Q'}\right)\\ &\leq \frac{W}{2} + Q' + \frac{x}{4(1-\rho)Q'} + \frac{x}{2W(1-\rho)}\\ &\leq \frac{x}{|\delta q|} + \frac{W}{2} + \frac{x}{4(1-\rho)Q} + \frac{x}{2(1-\rho)W}.\end{aligned}$$

□

5. Totals

Let x be given. We will choose U, V, W later; assume from the start that $U < x/2e$, $2 \cdot 10^6 \leq V < x/4$ and $UV \leq x$. Starting in section 5.2, we will also assume that $x \geq x_0 = (2e)^{2/5} \cdot 10^{25} = 6.891459\ldots \cdot 10^{26}$.

Let $\alpha \in \mathbb{R}/\mathbb{Z}$ be given. We choose an approximation $\alpha = a/q + \delta/x$, $\gcd(a, q) = 1$, $q \leq Q$, $|\delta/x| \leq 1/qQ$. We assume $Q \geq \sqrt{ex}$ and $Q \geq \max(4U, x/U)$. Let $S_{I,1}$, $S_{I,2}$, S_{II}, S_0 be as in (2.16), with the smoothing function $\eta = \eta_0$ as in (1.4).

The term S_0 is 0 because $V < x/4$ and η_0 is supported on $[-1/4, 1]$. Consider $w = 1$ and $w = 2$.

5.1. **Contributions of different types.**

5.1.1. *Type I terms:* $S_{I,1}$. The term $S_{I,1}$ can be handled directly by Lemma 3.4, with $\rho_0 = 4$. (Condition (3.11) is valid thanks to (2.4).) We will always have

$Q/4 > U$. Thus the contribution of $S_{I,1}$ is bounded by (3.12) and (3.13):
(5.1)

$$\begin{aligned}S_{I,1} &= \frac{x\mu(q)}{q}\frac{\phi(w/q^+)}{w/q^+}\widehat{\eta_0}(-\delta)\cdot \sum_{\substack{m\leq M_0/q^-\\ \gcd(m,wq^-)=1}} \frac{\mu(m)}{m}\log\frac{x/q}{m}\\ &+ \frac{x\mu(q)}{q}\left(\frac{\phi(w/q^+)}{w/q^+}\widehat{\log\cdot\eta_0}(-\delta) - \sum_{d|\frac{w}{q^+}}\frac{\mu(d)\log d}{d}\widehat{\eta_0}(-\delta)\right)\cdot \sum_{\substack{m\leq M_0/q^-\\ \gcd(m,wq^-)=1}}\frac{\mu(m)}{m}\\ &+ O_\leq\left(\left(\frac{q}{\pi}\log 2eq\log x + \frac{Uw}{\pi}\left(\log\frac{16cex}{Uw}\log\frac{ex}{Uw}+1\right)\right)\cdot 8\log 2\right)\\ &+ O_\leq\left(\left(\frac{2\tau(w)}{e} + \sigma(w)\frac{U^2 w}{xq}\log\frac{e^{1/2}x}{U}\right)\cdot\left(\frac{1}{8}-\frac{1}{2\pi^2}\right)\cdot 48\right),\end{aligned}$$

where we use (2.3), and where we set $M_0 = \min(U, \lfloor x/(|\delta q|)\rfloor/w)$, $q^+ = \gcd(q,w)$, $q^- = q/q^+$ and $c = 1 + 1/(4\log 2)$. Note that $M_0 = U$ (because $|\delta q|/x \leq 1/Q$ and $Q/2w > U$) and

$$\sum_{d|w/q^+}\frac{\mu(d)\log d}{d} = \begin{cases} 0 & \text{if } w=1 \text{ or } q \text{ is even,}\\ -\frac{\log 2}{2} & \text{if } w=2 \text{ or } q \text{ is odd.}\end{cases}$$

Recall (Ramaré; (2.8)) that

$$\sum_{\substack{m\leq U/q^-\\ \gcd(m,wq^-)=1}}\frac{\mu(m)}{m}\log\frac{U/q^-}{m} = O_\leq(1.4 + wq^-/\phi(wq^-)). \tag{5.2}$$

Let

$$\eta_1(t) = \log\left(\frac{x}{q^+U}t\right)\eta_0(t)$$

By (2.2, (2.4) and the fact that $x/U > 4$,

$$\begin{aligned}|\widehat{\eta_0}(-\delta)| &\leq O_\leq\left(\min\left(1,\frac{48}{(2\pi\delta)^2}\right)\right)\\ |\widehat{\eta_1}(-\delta)| &\leq O_\leq\left(\min\left(1,\frac{48}{(2\pi\delta)^2}\right)\right)\cdot\log\frac{x}{q^+U}.\end{aligned} \tag{5.3}$$

Now

$$\widehat{\eta_1}(t) = \widehat{\log\cdot\eta_0} + \log\frac{x}{q^+U}\widehat{\eta_0} = \widehat{\log\cdot\eta_0} - \log\frac{x/q}{U/q^-}\widehat{\eta_0}.$$

Hence the first two lines of (5.1) equal

$$\begin{aligned}&\frac{x\mu(q)}{q}\frac{\phi(w/q^+)}{w/q^+}\widehat{\eta_0}(-\delta)\cdot\sum_{\substack{m\leq U/q^-\\ \gcd(m,wq^-)=1}}\frac{\mu(m)}{m}\log\frac{U/q^-}{m}\\ &+\frac{x\mu(q)}{q}\left(\frac{\phi(w/q^+)}{w/q^+}\widehat{\eta_1}(-\delta) - \sum_{d|w/q^+}\frac{\mu(d)\log(d)}{d}\widehat{\eta_0}(-\delta)\right)\cdot\sum_{\substack{m\leq U/q^-\\ \gcd(m,wq^-)=1}}\frac{\mu(m)}{m}\end{aligned}$$

By (2.5), (2.7) and (5.2), the absolute value of this is bounded by

$$\frac{x}{q}\min\left(1,\frac{48}{(2\pi\delta)^2}\right)\left(1.4+\frac{wq^-}{\phi(wq^-)}+\left(\log\frac{x}{q^+U}\right)\lambda_1\right) \tag{5.4}$$

if $w=1$ or q is even, and by

$$\frac{x}{q}\min\left(1,\frac{48}{(2\pi\delta)^2}\right)\left(\frac{1}{2}\left(1.4+\frac{wq}{\phi(wq)}\right)+\frac{1}{2}\left(\log\frac{2x}{q^+U}\right)\lambda_1\right) \tag{5.5}$$

if $w=2$ and q is odd, where

$$\lambda_1=\begin{cases}\frac{2q/\phi(q)}{\log\frac{U}{w(q^-)^2}} & \text{if } w(q^-)^2\leq U/e^{2q/\phi(q)},\\ 1 & \text{otherwise.}\end{cases} \tag{5.6}$$

Since (a) $\log 2<1.4$ and (b) $wq^-/\phi(wq^{-1})=q/\phi(q)$ if $w=1$ or q is even, (c) $wq^-/\phi(wq^{-1})=2q/\phi(q)$ if $w=2$ and q is odd, (5.4) and (5.5) are both bounded by

$$\frac{x}{q}\min\left(1,\frac{48}{(2\pi\delta)^2}\right)\left(1.4+\frac{q}{\phi(q)}+\frac{q^+}{w}\left(\log\frac{x}{q^+U}\right)\lambda_1\right). \tag{5.7}$$

We obtain

$$\begin{aligned}|S_{I,1}|\leq{}&\frac{x}{q}\min\left(1,\frac{48}{(2\pi\delta)^2}\right)\left(1.4+\frac{q}{\phi(q)}+\frac{q^+}{w}\left(\log\frac{x}{q^+U}\right)\lambda_1\right)\\ &+1.7651q\log 2eq\log x+1.7651Uw\left(\log\frac{59.1792x}{Uw}\log\frac{ex}{Uw}+1\right)\\ &+3.5683\left(\sigma(w)\frac{U^2w}{xq}\log\frac{e^{1/2}x}{U}+\frac{2\tau(w)}{e}\right),\end{aligned} \tag{5.8}$$

where λ_1 is as in (5.6).

5.1.2. *Type I terms: $S_{I,2}$. The case $q\leq Q/V$.* If $q\leq Q/V$, then, for $v\leq V$,

$$v\alpha=\frac{va}{q}+O_{\leq}\left(\frac{v}{Qq}\right)=\frac{va}{q}+O_{\leq}\left(\frac{1}{q^2}\right),$$

and so va/q is a valid approximation to $d\alpha$. We can thus estimate $S_{I,2}$ by applying Lemma 3.3 to each inner double sum in $S_{I,2}$ (2.16). We obtain

(5.9)
$$\begin{aligned}
S_{I,2} = &\sum_{\substack{v\leq V\\ \gcd(v,w)=1}} \Lambda(v) \sum_{\substack{m\leq U\\ \gcd(m,w)=1}} \mu(m) \sum_{\substack{n\\ \gcd(n,w)=1}} e((v\alpha)\cdot mn)\eta(mn/(x/v))\\
= &\sum_{\substack{v\leq V\\ \gcd(v,w)=1}} \Lambda(v)\frac{x\mu(q_v)}{vq_v}\frac{\phi(w/q_v^+)}{w/q_v^+}\widehat{\eta}(-\delta)\cdot \sum_{\substack{m\leq M_v/q_v^-\\ \gcd(m,wq_v^-)=1}} \frac{\mu(m)}{mv}\\
+ &\sum_{\substack{v\leq V\\ \gcd(v,w)=1}} \Lambda(v)\cdot O_{\leq}\left(\frac{U(U+q_v)\sigma(w)}{xq_v/vw}\cdot\left(\frac{1}{8}-\frac{1}{2\pi^2}\right)|\eta''|_1\right)\\
+ &\sum_{\substack{v\leq V\\ \gcd(v,w)=1}} \Lambda(v)|\eta'|_1\cdot\left(\frac{q_v}{\pi}\log 2eq_v + \frac{Uw}{\pi}\left(2(1+\epsilon)\log\frac{2ecx}{Uv}+\log\frac{16ecx}{Uvw}\right)\right)\\
+ &\sum_{\substack{v\leq V\\ \gcd(v,w)=1}} \Lambda(v)(2|\eta|_1+|\eta'|_1)\cdot\left(\frac{3wcx/v}{Q/v}+\frac{3(1+\epsilon)}{2\epsilon}\frac{x/v}{Q/v}\log\frac{Uw}{M_v}\right),
\end{aligned}$$

where we write $q_v = q/\gcd(v,q)$, $q_v^+ = \gcd(q_v,w)$, $q_v^- = q_v/q_v^+$, $c = 2|\eta|_1/|\eta'|_1 + 1$ and $M_v = \max(\min(U,Q/2wv),1)$. We bound the first sum as in (3.17), using (2.7); we obtain

$$\begin{aligned}
&\left|\sum_{\substack{v\leq V\\ \gcd(v,w)=1}} \Lambda(v)\frac{x\mu(q_v)}{vq_v}\frac{\phi(w/q_v^+)}{w/q_v^+}\widehat{\eta}(-\delta)\cdot \sum_{\substack{m\leq M/q_v^-\\ \gcd(m,wq_v^-)=1}} \frac{\mu(m)}{m}\right|\\
&\leq \frac{x\widehat{\eta}(-\delta)}{q}\sum_{\substack{v\leq V\\ \gcd(v,w)=1}} \Lambda(v)\frac{\gcd(q,v)}{v}\frac{\phi(w/q_v^+)}{w/q_v^+}\min\left(2\frac{wq_v^-/\phi(wq_v^-)}{\log\frac{M_v}{w(q_v^-)^2}},1\right)\\
&\leq \frac{x\widehat{\eta}(-\delta)}{q}\lambda_2\log(Vq)
\end{aligned}$$

by (2.10), where

$$\lambda_2 = \begin{cases} \frac{2q/\phi(q)}{\log\frac{M_V}{w(q^-)^2}} & \text{if } w(q^-)^2\leq M_V/e^{2q/\phi(q)},\\ 1 & \text{otherwise.}\end{cases} \tag{5.10}$$

The second sum is at most

(5.11)
$$\left(\frac{1}{8}-\frac{1}{2\pi^2}\right)\sigma(w)w\frac{U}{qx}|\eta''|_1\cdot\left(\sum_{v\leq V}\Lambda(v)(U+q)v + \sum_{\substack{v\leq V\\ \gcd(v,q)\neq 1}}\Lambda(v)v\cdot U\gcd(v,q)\right).$$

By (2.13),

$$\sum_{v\leq V} \Lambda(v)(U+q)v < 1.03884\frac{V^2}{2}(U+q).$$

Proceeding a little coarsely,

$$\begin{aligned}\sum_{\substack{v\leq V\\ \gcd(v,q)=1}} \Lambda(v)v\cdot\gcd(v,q) &\leq \sum_{p|q}\sum_{\alpha:p^\alpha\leq V}(\log p)\cdot p^\alpha\cdot p^{\min(\alpha,v_p(q))}\\ &\leq \sum_{p|q}(\log p)\cdot p^{\lfloor\log V/\log q\rfloor}p^{v_p(q)} \leq 2Vq\log q.\end{aligned}$$

Thus (5.11) is at most

$$\sigma(w)w\left(1.8535\frac{U(U+q)}{qx}V^2 + 7.137\frac{U^2V}{x}\log q\right).$$

We can use (2.12) and partial summation to bound

$$\begin{aligned}&\sum_{v\leq V}\Lambda(v)\left(2(1+\epsilon)\log\frac{2ecx}{Uv}+\log\frac{16ecx}{Uvw}\right)\\ &\quad\leq 1.03883V\left(2(1+\epsilon)\log\frac{2e^2cx}{UV}+\log\frac{16e^2cx}{UVw}\right).\end{aligned}$$

We bound other error terms in (5.9) using (2.11), bounding q_v from above by q when needed.

We obtain

(5.12)

$$\begin{aligned}|S_{I,2}| \leq\ &\frac{x\widehat{\eta}(-\delta)}{q}\lambda_2\log(Vq) + 1.8337UVw\left(2(1+\epsilon)\log\frac{2e^2cx}{UV}+\log\frac{16e^2cx}{UVw}\right)\\ &+\sigma(w)w\left(1.8535\frac{U(U+q)}{qx}V^2+7.137\frac{U^2V}{x}\log q\right)\\ &+1.7658q\log 2eq + 30.8119\frac{wx}{Q/V} + 11.3223\frac{x(1+\epsilon)}{Q\epsilon/V}\log\frac{2UVw^2}{Q}.\end{aligned}$$

where $M_V = \max(\min(U, Q/2wqV), 1)$. We will choose $\epsilon > 0$ later.

The case $q > Q/V$. Lemma 3.5 gives us (3.14) and (3.15), which we use directly.

5.1.3. *Type II terms.* The quantity to bound here is (4.4), that is,

$$4\int_V^{x/U}\sqrt{S_1(U,W)\cdot S_2(U,V,W)}\frac{dW}{W} + 4\int_V^{x/U}\sqrt{S_1(U,W)\cdot S_3(W)}\frac{dW}{W}, \tag{5.13}$$

where S_1, S_2 and S_3 are as in (4.5) and (4.6). We bounded S_1 in (4.33) and (4.34), S_2 in Prop. 4.5 and S_3 in (4.6).

We first recall our estimate for S_1. In the whole range $[V, x/U]$ for W, we know from (4.33) and (4.34) that $S_1(U,W)$ is at most

$$\frac{3}{\pi^2}\frac{x}{W} + \kappa_{1,0}\zeta(3/2)^3\frac{x}{W}\sqrt{\frac{x/WU}{U}} \tag{5.14}$$

if $w = 1$,

$$\frac{2}{\pi^2}\frac{x}{W} + \kappa_{2,0}\zeta(3/2)^3\frac{x}{W}\sqrt{\frac{x/WU}{U}} \tag{5.15}$$

if $w = 2$, where

$$\kappa_{w,0} = \begin{cases} 5.08 & \text{if } w = 1,\\ 1.27 & \text{if } w = 2.\end{cases}$$

We have better estimates for the constant in front in some parts of the range; in what is usually the main part, (4.33) and (4.34) give us constants of 0.22125 and 0.15107 instead of $3/\pi^2$ and $2/\pi^2$. Note $5.08\zeta(3/2)^3 = 90.5670\dots$ and $1.27\zeta(3/2)^3 = 22.6417\dots$. We should choose U, V so that the first term dominates. For the while being, assume only

$$U \geq 5 \cdot 10^5 \frac{x}{VU}; \tag{5.16}$$

then (5.14) and (5.15) give

$$S_1(U, W) \leq \kappa_{w,1} \frac{x}{W}, \tag{5.17}$$

where

$$\kappa_{w,1} = \begin{cases} \frac{3}{\pi^2} + \frac{90.5671}{\sqrt{10^6/2}} \leq 0.4321 & \text{if } w = 1,\\ \frac{2}{\pi^2} + \frac{22.6418}{\sqrt{10^6/2}} \leq 0.2347 & \text{if } w = 2,\end{cases}$$

and this will suffice for our cruder estimates.

The second integral in (5.13) is now easy to bound. By (4.6),

$$S_3(W) \leq 1.0171x + 2.0341W \leq 1.0172x,$$

since $W \leq x/U \leq x/10^6$. Hence

$$\begin{aligned} 4\int_V^{x/U} \sqrt{S_1(U,W) \cdot S_3(W)}\, \frac{dW}{W} &\leq 4\int_V^{x/U} \sqrt{\kappa_{w,1}\frac{x}{W} \cdot 1.0172x}\, \frac{dW}{W}\\ &\leq \kappa_{w,9} \frac{x}{\sqrt{V}},\end{aligned}$$

where

$$\kappa_{w,9} = 8 \cdot \sqrt{1.0172 \cdot \kappa_{w,1}} \leq \begin{cases} 5.3035 & \text{if } w = 1,\\ 3.9086 & \text{if } w = 2.\end{cases}$$

Let us now examine S_2, which was bounded in Prop. 4.5. Recall $W' = \max(V, W/2)$, $U' = \max(U, x/2W)$. Since $W' \geq W/2$ and $W \geq V \geq 117$, we can always bound

$$\sum_{W' < p \leq W} (\log p)^2 \leq \frac{1}{2} W (\log W). \tag{5.18}$$

by (2.14).

Bounding S_2 for δ arbitrary. We set

$$W_0 = \min(\max(2\theta q, V), x/U),$$

where $\theta \geq e$ is a parameter that will be set later.

For $V \leq W < W_0$, we use the bound (4.48):

$$\begin{aligned} S_2(U', W', W) &\leq \left(\max(1, 2\rho)\left(\frac{x}{4q} + \frac{x}{W}\right) + \frac{W}{2} + 2q\right) \cdot \frac{1}{2} W (\log W)\\ &\leq \max\left(\frac{1}{2}, \rho\right)\left(\frac{W}{4q} + 1\right) x \log W + \frac{W^2 \log W}{4} + qW \log W,\end{aligned}$$

where $\rho = q/Q$.

If $W_0 > V$, the contribution of the terms with $V \leq W < W_0$ to (5.13) is (by 5.17) bounded by

$$\begin{aligned}
&4\int_V^{W_0} \sqrt{\kappa_{w,1}\frac{x}{W}\left(\frac{\rho_0}{2}\left(\frac{W}{4q}+1\right)x\log W + \frac{W^2\log W}{4} + qW\log W\right)}\,\frac{dW}{W} \\
&\leq \frac{\kappa_{w,2}}{2}\sqrt{\rho_0}x\int_V^{W_0}\frac{\sqrt{\log W}}{W^{3/2}}dW + \frac{\kappa_{w,3}}{2}\sqrt{x}\int_V^{W_0}\frac{\sqrt{\log W}}{W^{1/2}}dW \\
&+ \kappa_{w,3}\sqrt{\frac{\rho_0 x^2}{8q}+qx}\int_V^{W_0}\frac{\sqrt{\log W}}{W}dW \qquad (5.19)\\
&\leq \left(\kappa_{w,2}\sqrt{\rho_0}\frac{x}{\sqrt{V}} + \kappa_{w,3}\sqrt{xW_0}\right)\sqrt{\log W_0} \\
&+ \frac{\kappa_{w,3}}{3}\sqrt{\frac{\rho_0 x^2}{8q}+qx}\left((\log W_0)^{3/2} - (\log V)^{3/2}\right),
\end{aligned}$$

where $\rho_0 = \max(1, 2\rho)$,

$$\kappa_{w,2} = 4\sqrt{2\kappa_{w,1}} \leq \begin{cases} 3.7183 & \text{if } w = 1, \\ 2.7403 & \text{if } w = 2 \end{cases}$$

and

$$\kappa_{w,3} = 4\sqrt{\kappa_{w,1}} \leq \begin{cases} 2.62921 & \text{if } w = 1, \\ 1.93768 & \text{if } w = 2. \end{cases}$$

We now examine the terms with $W \geq W_0$. (If $\theta q > x/U$, then $W_0 = U/x$, the contribution of the case is nil, and the computations below can be ignored.)

We use (4.49):

$$S_2(U', W', W) \leq \left(\frac{x}{2\phi(q)}\frac{1}{\log(W/2q)} + \frac{q}{\phi(q)}\frac{W}{\log(W/2q)}\right)\cdot\frac{1}{2}W\log W.$$

By $\sqrt{a+b} \leq \sqrt{a} + \sqrt{b}$, we can take out the $q/\phi(q) \cdot W/\log(W/2q)$ term and estimate its contribution on its own; it is at most

$$\begin{aligned}
&4\int_{W_0}^{x/U}\sqrt{\kappa_{w,1}\frac{x}{W}\cdot\frac{q}{\phi(q)}\cdot\frac{1}{2}W^2\frac{\log W}{\log W/2q}}\,\frac{dW}{W} \\
&= \kappa_{w,2}\sqrt{\frac{q}{\phi(q)}}\int_{W_0}^{x/U}\sqrt{\frac{x\log W}{W\log W/2q}}dW \qquad (5.20)\\
&\leq \kappa_{w,2}\sqrt{\frac{qx}{\phi(q)}}\int_{W_0}^{x/U}\frac{1}{\sqrt{W}}\left(1+\sqrt{\frac{\log 2q}{\log W/2q}}\right)dW
\end{aligned}$$

Now

$$\int_{W_0}^{x/U}\frac{1}{\sqrt{W}}\sqrt{\frac{\log 2q}{\log W/2q}}dW \leq \sqrt{2q\log 2q}\int_{\max(\theta,V/2q)}^{x/2Uq}\frac{1}{\sqrt{t\log t}}dt.$$

We bound this last integral somewhat crudely: for $T \geq e$,

$$\int_e^T \frac{1}{\sqrt{t\log t}}dt \leq 2.3\sqrt{\frac{T}{\log T}} \qquad (5.21)$$

(by numerical work for $e \le T \le T_0$ and by comparison of derivatives for $T > T_0$, where $T_0 = e^{(1-2/2.3)^{-1}} = 2135.94\dots$). Since $\theta \ge e$, this gives us that

$$\int_{W_0}^{x/U} \frac{1}{\sqrt{W}} \left(1 + \sqrt{\frac{\log 2q}{\log W/2q}}\right) dW$$
$$\le 2\sqrt{\frac{x}{U}} + 2.3\sqrt{2q\log 2q} \cdot \sqrt{\frac{x/2Uq}{\log x/2Uq}},$$

and so (5.20) is at most

$$2\kappa_{w,2}\sqrt{\frac{q}{\phi(q)}} \left(1 + 1.15\sqrt{\frac{\log 2q}{\log x/2Uq}}\right) \frac{x}{\sqrt{U}}.$$

We are left with what will usually be the main term, viz.,

$$4\int_{W_0}^{x/U} \sqrt{S_1(U,W) \cdot \left(\frac{x}{4\phi(q)} \frac{\log W}{\log W/2q}\right) W} \frac{dW}{W}, \tag{5.22}$$

which, by (4.33) and (4.34), is at most $x/\sqrt{\phi(q)}$ times the integral of

$$\frac{1}{W}\sqrt{\left(4H_w\left(\frac{x}{WU}\right) + \kappa_{w,4}\sqrt{\frac{x/WU}{U}}\right) \frac{\log W}{\log W/2q}}$$

for W going from W_0 to x/U, where H_w is as in (4.35) and

$$\kappa_{w,4} = 4\kappa_{w,0}\zeta(3/2)^3 \le \begin{cases} 362.2684 & \text{if } w = 1, \\ 90.5671 & \text{if } w = 2. \end{cases}$$

By the arithmetic/geometric mean inequality, the integrand is at most $1/W$ times

$$\frac{\beta + \beta^{-1} \cdot 4H_w(x/WU)}{2} + \frac{\beta^{-1}}{2}\kappa_{w,4}\sqrt{\frac{x/WU}{U}} + \frac{\beta}{2}\frac{\log 2q}{\log W/2q} \tag{5.23}$$

for any $\beta > 0$. We will choose β later.

The first summand in (5.23) gives what we can think of as the main or worst term in the whole paper; let us compute it first. The integral is

$$\begin{aligned} \int_{W_0}^{x/U} \frac{\beta + \beta^{-1} \cdot 4H_w(x/WU)}{2} \frac{dW}{W} &= \int_1^{x/UW_0} \frac{\beta + \beta^{-1} \cdot 4H_w(s)}{2} \frac{ds}{s} \\ &\le \begin{cases} \left(\frac{\beta}{2} + \frac{\kappa_{w,6}}{2\beta}\right) \log \frac{x}{UW_0} & \text{if } w = 1, \\ \left(\frac{\beta}{2} + \frac{\kappa_{w,6}}{2\beta}\right) \log \frac{x}{UW_0} & \text{if } w = 2 \end{cases} \end{aligned} \tag{5.24}$$

by (4.36), where

$$\kappa_{w,6} = \begin{cases} 0.89928 & \text{if } w = 1, \\ 0.60428 & \text{if } w = 2. \end{cases}$$

Thus the main term is simply

$$\left(\frac{\beta}{2} + \frac{\kappa_{w,6}}{2\beta}\right) \frac{x}{\sqrt{\phi(q)}} \log \frac{x}{UW_0}. \tag{5.25}$$

The integral of the second summand is at most

$$\beta^{-1} \cdot \frac{\kappa_{w,4}}{2} \frac{\sqrt{x}}{U} \int_V^{x/U} \frac{dW}{W^{3/2}} \le \beta^{-1} \cdot \kappa_{w,4}\sqrt{\frac{x/UV}{U}}.$$

By (5.16), this is at most

$$\beta^{-1}\sqrt{2}\cdot 10^{-3}\cdot \kappa_{w,4} \leq \beta^{-1}\kappa_{w,7},$$

where

$$\kappa_{w,7} = \frac{\sqrt{2}\kappa_{w,4}}{1000} \leq \begin{cases} 0.5124 & \text{if } w=1,\\ 0.1281 & \text{if } w=2.\end{cases}$$

Thus the contribution of the second summand is at most

$$\beta^{-1}\kappa_{w,7}\cdot\frac{x}{\sqrt{\phi(q)}}.$$

The integral of the third summand is

$$\frac{\beta}{2}\int_{W_0}^{x/U}\frac{\log 2q}{\log W/2q}\frac{dW}{W}. \tag{5.26}$$

If $V < 2\theta q \leq x/U$, this is

$$\begin{aligned}\frac{\beta}{2}\int_{2\theta q}^{x/U}\frac{\log 2q}{\log W/2q}\frac{dW}{W} &= \frac{\beta}{2}\log 2q\cdot\int_{\theta}^{x/2Uq}\frac{1}{\log t}\frac{dt}{t}\\ &= \frac{\beta}{2}\log 2q\cdot\left(\log\log\frac{x}{2Uq} - \log\log\theta\right).\end{aligned}$$

If $2\theta q > x/U$, the integral is over an empty range and its contribution is hence 0.

If $2\theta q \leq V$, (5.26) is

$$\begin{aligned}\frac{\beta}{2}\int_{V}^{x/U}\frac{\log 2q}{\log W/2q}\frac{dW}{W} &= \frac{\beta\log 2q}{2}\int_{V/2q}^{x/2Uq}\frac{1}{\log t}\frac{dt}{t}\\ &= \frac{\beta\log 2q}{2}\cdot\left(\log\log\frac{x}{2Uq} - \log\log V/2q\right)\\ &= \frac{\beta\log 2q}{2}\cdot\log\left(1+\frac{\log x/UV}{\log V/2q}\right).\end{aligned}$$

(Of course, $\log(1 + (\log x/UV)/(\log V/2q)) \leq (\log x/UV)/(\log V/2q)$; this is smaller than $(\log x/UV)/\log 2q$ when $V/2q > 2q$.)

The total bound for (5.22) is thus

$$\frac{x}{\sqrt{\phi(q)}}\cdot\left(\beta\cdot\left(\frac{1}{2}\log\frac{x}{UW_0}+\frac{\Phi}{2}\right)+\beta^{-1}\left(\frac{1}{2}\kappa_{w,6}\log\frac{x}{UW_0}+\kappa_{w,7}\right)\right), \tag{5.27}$$

where

$$\Phi = \begin{cases}\log 2q\left(\log\log\frac{x}{2Uq} - \log\log\theta\right) & \text{if } V/2\theta < q < x/(2\theta U).\\ \log 2q\log\left(1+\frac{\log x/UV}{\log V/2q}\right) & \text{if } q \leq V/2\theta.\end{cases} \tag{5.28}$$

Choosing β optimally, we obtain that (5.22) is at most

$$\frac{x}{\sqrt{\phi(q)}}\sqrt{\left(\log\frac{x}{UW_0}+\Phi\right)\left(\kappa_{w,6}\log\frac{x}{UW_0}+2\kappa_{w,7}\right)}, \tag{5.29}$$

where Φ is as in (5.28).

Bounding S_2 for $|\delta| \geq 4$. Let us see how much a non-zero δ can help us. It makes sense to apply (4.51) only when $|\delta| \geq 4$; otherwise (4.49) is almost certainly better. Now, by definition, $|\delta|/x \leq 1/qQ$, and so $|\delta| \geq 4$ can happen only when $q \leq x/2Q$.

With this in mind, let us apply (4.51). Note first that

$$\frac{x}{|\delta q|}\left(q+\frac{x}{2W}\right)^{-1} \geq \frac{2/|\delta q|}{\frac{1}{Q}+\frac{1}{W}} \geq \frac{\min(Q,W)}{|\delta q|}.$$

Recalling also (5.18), we see that (4.51) gives us

$$\begin{aligned} S_2(U',W',W) \leq \min&\left(1, \frac{2q/\phi(q)}{\log\frac{\min(Q,W)}{|\delta q|}}\right) \\ &\cdot\left(\frac{x}{|\delta q|}+\frac{W}{2}+\frac{x}{4(1-\rho)Q}+\frac{x}{2(1-\rho)W}\right)\cdot\frac{1}{2}W(\log W),\end{aligned}$$

where $\rho = q/Q$. Note also that $|\delta q| \leq x/Q$.

The contribution (to (5.13)) of the terms $W/2$ and $x/(4(1-\rho)Q)$ can be set aside (by $\sqrt{a+b}\leq\sqrt{a}+\sqrt{b}$) and bounded somewhat crudely; it is at most

$$\begin{aligned} 2\sqrt{\kappa_{w,1}}\sqrt{x}\int_V^{x/U} &\sqrt{\frac{1}{W}\frac{W+\frac{x}{2(1-\rho)Q}}{\max\left(1,\frac{1}{2}\frac{\phi(q)}{q}\log\frac{\min(Q,W)}{|\delta q|}\right)}\cdot W\log W}\frac{dW}{W} \\ &\leq 2\sqrt{\kappa_{w,1}}\left(2\frac{x}{\sqrt{U}}+\frac{x}{\sqrt{2(1-\rho)Q}}\log\frac{x}{UV}\right) \\ &\cdot \max_{V\leq W\leq\frac{X}{U}}\sqrt{\frac{\log W}{\max\left(1,\frac{1}{2}\frac{\phi(q)}{q}\log\frac{\min(Q,W)}{|\delta q|}\right)}}. \end{aligned}\tag{5.30}$$

Since $Q\geq x/U$, $\min(Q,W)=W$ for all $W\leq x/U$. Since $t\mapsto(\log t)/(\log t/c)$ is decreasing for $t>c$, it follows that

$$\max_{V\leq W\leq\frac{X}{U}}\sqrt{\frac{\log W}{\frac{1}{2}\frac{\phi(q)}{q}\log\frac{\min(Q,W)}{|\delta q|}}} \leq \sqrt{\frac{2q}{\phi(q)}\frac{\log V}{\log\frac{V}{|\delta q|}}};$$

if $|\delta q|\leq V$ (as is the necessarily the case if $Q\geq ex/V$). However, if $|\delta q|>V/\theta$, we use the bound

$$\max_{V\leq W\leq\frac{X}{U}}\sqrt{\frac{\log W}{\max\left(1,\frac{1}{2}\frac{\phi(q)}{q}\log\frac{\min(Q,W)}{|\delta q|}\right)}} \leq \frac{\sqrt{\log\theta|\delta q|}}{\sqrt{\min\left(1,\frac{1}{2}\frac{\phi(q)}{q}\log\theta\right)}}.$$

The contribution of $x/2(1-\rho)W$ can also be set aside; since in it the values of W close to V dominate, we choose to bound the contribution by

$$\begin{aligned} 4\int_V^{x/U}&\sqrt{\kappa_{w,1}\frac{x}{W}\cdot\left(\frac{x}{2(1-\rho)W}\right)\cdot\frac{1}{2}W\log W}\frac{dW}{W} \\ &= 4\sqrt{\kappa_{w,1}}\frac{x}{2\sqrt{1-\rho}}\int_V^{x/U}\frac{\sqrt{\log W}}{W^{3/2}}dW \\ &\leq \frac{2\sqrt{\kappa_{w,1}}}{\sqrt{1-\rho}}x\int_V^{\infty}\frac{\sqrt{\log W}}{W^{3/2}}dW \\ &\leq 4\frac{\sqrt{\kappa_{w,1}}}{\sqrt{(1-\rho)V}}\left(\sqrt{\log V}+\sqrt{1/\log V}\right)x, \end{aligned}$$

where we use the estimate

$$\begin{aligned}\int_V^\infty \frac{\sqrt{\log W}}{W^{3/2}} dW &= \frac{1}{\sqrt{V}} \int_1^\infty \frac{\sqrt{\log u + \log V}}{u^{3/2}} du \\ &\leq \frac{1}{\sqrt{V}} \int_1^\infty \frac{\sqrt{\log V}}{u^{3/2}} du + \frac{1}{\sqrt{V}} \int_1^\infty \frac{1}{2\sqrt{\log V}} \frac{\log u}{u^{3/2}} du \\ &= 2\frac{\sqrt{\log V}}{\sqrt{V}} + \frac{1}{2\sqrt{V \log V}} \cdot 4 \leq \frac{2}{\sqrt{V}} \left(\sqrt{\log V} + \sqrt{1/\log V}\right).\end{aligned}$$

What is left to estimate then is the main term, coming from $x/q\delta$. Similarly to before, we define $W_0 = \max(V, \theta|\delta q|)$, where $\theta \geq e$ will be set later. We bound the main term by $2x/\sqrt{\delta \phi(q)}$ times the integral of

$$\frac{1}{W}\sqrt{\left(H_w\left(\frac{x}{WU}\right) + \kappa_{w,4}\sqrt{\frac{x/WU}{U}}\right) \frac{\log W}{\log \frac{W}{|\delta q|}}} \tag{5.31}$$

from W_0 to x/U, plus (if $W_0 > W$) the integral

$$4\int_V^{\theta|\delta q|} \sqrt{S_1(U,W) \cdot \frac{x}{|\delta q|} \cdot \frac{1}{2} W \log W} \frac{dW}{W}. \tag{5.32}$$

This contribution of (5.31) is bounded (much as in (5.24)) by

$$\frac{2x}{\sqrt{|\delta|\phi(q)}} \sqrt{\left(\log \frac{x}{UW_0} + \Phi\right)\left(\kappa_{w,6} \log \frac{x}{UW_0} + 2\kappa_{w,7}\right)},$$

where

$$\Phi = \begin{cases} \log|\delta q| \log\left(1 + \frac{\log x/UV}{\log V/|\delta| q}\right) & \text{if } |\delta q| \leq V/\theta, \\ \log|\delta q| \left(\log\log \frac{x}{U|\delta q|} - \log\log\theta\right) & \text{if } V/\theta < |\delta q| \leq x/\theta U. \end{cases} \tag{5.33}$$

We have $W_0 > W$ if and only if $\theta|\delta q| > V$; in that case, we must also consider the integral (5.32), which we bound as follows:

$$\begin{aligned}4\int_V^{\theta|\delta q|} \sqrt{S_1(U,W) \cdot \frac{x}{|\delta q|} \cdot \frac{1}{2} W \log W} \frac{dW}{W} &\leq \sqrt{8\kappa_{w,1}} \frac{x}{\sqrt{|\delta q|}} \int_V^{\theta|\delta q|} \frac{\sqrt{\log W}}{W} dW \\ &= \frac{\kappa_{w,2}}{6} \frac{x}{\sqrt{|\delta q|}} \left((\log\theta|\delta q|)^{3/2} - (\log V)^{3/2}\right).\end{aligned}$$

* * *

It is time to collect all type II terms. Let us start with the case of general δ. We will set $\theta \geq e$ later. If $q \leq V/2\theta$, then $|S_{II}|$ is at most

$$\begin{aligned}&\frac{x}{\sqrt{\phi(q)}} \cdot \sqrt{\left(\log \frac{x}{UV} + \log 2q \log\left(1 + \frac{\log x/UV}{\log V/2q}\right)\right)\left(\kappa_{w,6} \log \frac{x}{UV} + 2\kappa_{w,7}\right)} \\ &+ 2\kappa_{w,2}\sqrt{\frac{q}{\phi(q)}}\left(1 + 1.15\sqrt{\frac{\log 2q}{\log x/2Uq}}\right)\frac{x}{\sqrt{U}} + \kappa_{w,9}\frac{x}{\sqrt{V}}.\end{aligned} \tag{5.34}$$

If $V/2\theta < q \leq x/2\theta U$, then $|S_{II}|$ is at most

(5.35)
$$\begin{aligned}
&\frac{x}{\sqrt{\phi(q)}} \cdot \sqrt{\left(\log \frac{x}{U\cdot 2\theta q} + \log 2q \log \frac{\log x/2Uq}{\log \theta}\right)\left(\kappa_{w,6} \log \frac{x}{U\cdot 2\theta q} + 2\kappa_{w,7}\right)}\\
&+ 2\kappa_{w,2} \sqrt{\frac{q}{\phi(q)}} \left(1 + 1.15 \sqrt{\frac{\log 2q}{\log x/2Uq}}\right) \frac{x}{\sqrt{U}} + (\kappa_{w,2} \sqrt{\log 2\theta q} + \kappa_{w,9}) \frac{x}{\sqrt{V}}\\
&+ \frac{\kappa_{w,3}}{6\sqrt{2}} \left((\log 2\theta q)^{3/2} - (\log V)^{3/2}\right) \frac{x}{\sqrt{q}}\\
&+ \kappa_{w,3} \left(\sqrt{2\theta \cdot \log 2\theta q} + \frac{(\log 2\theta q)^{3/2} - (\log V)^{3/2}}{3}\right) \sqrt{qx},
\end{aligned}$$

where we use the fact that $Q \geq x/U$ (implying that $\rho_0 = \max(1, 2q/Q)$ equals 1 for $q \leq x/2U$). Finally, if $q > x/2\theta U$,

$$\begin{aligned}
|S_{II}| \leq (\kappa_{w,2} \sqrt{2 \log x/U} + \kappa_{w,9}) \frac{x}{\sqrt{V}} + \kappa_{w,3} \sqrt{\log x/U} \frac{x}{\sqrt{U}}\\
+ \frac{\kappa_{w,3}}{3} ((\log x/U)^{3/2} - (\log V)^{3/2}) \left(\frac{x}{2\sqrt{q}} + \sqrt{qx}\right).
\end{aligned} \tag{5.36}$$

Now let us examine the alternative bounds for $|\delta| \geq 4$. If $|\delta q| \leq V/\theta$, then $|S_{II}|$ is at most

(5.37)
$$\begin{aligned}
&\frac{2x}{\sqrt{|\delta|\phi(q)}} \sqrt{\left(\log \frac{x}{UV} + \log |\delta q| \log \left(1 + \frac{\log x/UV}{\log V/|\delta q|}\right)\right)\left(\kappa_{w,6} \log \frac{x}{UV} + 2\kappa_{w,7}\right)}\\
&+ \left(\frac{4\sqrt{\kappa_{w,1}}}{\sqrt{(1-\rho)}} \left(\sqrt{\log V} + \sqrt{1/\log V}\right) + \kappa_{w,9}\right) \frac{x}{\sqrt{V}}\\
&+ \kappa_{w,2} \sqrt{\frac{q}{\phi(q)}} \left(\frac{1}{\sqrt{U}} + \frac{1}{\sqrt{8(Q-q)}} \log \frac{x}{UV}\right) \cdot \sqrt{\frac{\log V}{\log V/|\delta q|}} \cdot x.
\end{aligned}$$

If $|\delta q| > V/\theta$, then $|S_{II}|$ is at most

(5.38)
$$\begin{aligned}
&\frac{2x}{\sqrt{|\delta|\phi(q)}} \sqrt{\left(\log \frac{x}{U\theta|\delta|q} + \log |\delta q| \log \frac{\log x/U|\delta q|}{\log \theta}\right)\left(\kappa_{w,6} \log \frac{x}{U\theta|\delta q|} + 2\kappa_{w,7}\right)}\\
&+ \frac{\kappa_{w,2}}{6} \frac{x}{\sqrt{|\delta q|}} \left((\log \theta|\delta q|)^{3/2} - (\log V)^{3/2}\right)\\
&+ \left(\frac{4\sqrt{\kappa_{w,1}}}{\sqrt{(1-\rho)}} \left(\sqrt{\log V} + \sqrt{1/\log V}\right) + \kappa_{w,9}\right) \frac{x}{\sqrt{V}}\\
&+ \frac{\kappa_{w,2}}{\sqrt{2}} \left(\frac{1}{\sqrt{U}} + \frac{1}{\sqrt{8(Q-q)}} \log \frac{x}{UV}\right) \cdot \frac{\sqrt{\log \theta|\delta q|}}{\sqrt{\min\left(1, \frac{1}{2}\frac{\phi(q)}{q} \log \theta\right)}} \cdot x,
\end{aligned}$$

where $\rho = q/Q$. (Note $|\delta| \leq x/Qq$ implies $\rho \leq x/4Q^2$, and so ρ will be very small and $Q - q$ will be very close to Q.)

5.2. **Adjusting parameters. Calculations.** We must bound the exponential sum $\sum_n \Lambda(n)e(\alpha n)\eta(n/x)$. By (2.15), it is enough to sum the bounds (5.8) for

$S_{I,1}$, (5.12) for $S_{I,2}$ (or (3.15) if $q > Q/V$), (5.34)–(5.38) for S_{II}. We will now see how it will be best to set U, V and other parameters.

Usually, the largest terms will be

$$1.8337UVw\left(2(1+\epsilon)\log\frac{2e^2cx}{UV}+\log\frac{16e^2cx}{UVw}\right) \tag{5.39}$$

(from (5.12), type I) and

(5.40)

$$\frac{x}{\sqrt{|\delta_0|\phi(q)}}\sqrt{\left(\log\frac{x}{UV}+\log|2\delta_1 q|\log\left(1+\frac{\log\frac{x}{UV}}{\log\frac{V}{|2\delta_1 q|}}\right)\right)\left(\kappa_{w,6}\log\frac{x}{UV}+2\kappa_{w,7}\right)}$$

(from (5.34) and (5.37), type II; here $\delta_0 = \max(1,|\delta|/4)$, $\delta_1 = 1$ if $|\delta| \leq 4$ and $\delta_1 = |\delta|/2$ if $|\delta| > 4$). For now, we will think of ϵ as being (say) 1/4.

We must choose UV. We simply[3] set $UV = x/t$, where

$$t = K\sqrt{q\delta_1},$$

and K will be optimised for the range of greatest concern ($q \sim 2\cdot 10^6$, $\phi(q) \sim 4\cdot 10^5$). What we must minimize is x times

$$\frac{w}{t}(23.07416 - 1.8337\log w + 6.41795\log t)$$
$$+\frac{1}{\sqrt{\phi(q)\delta_0}}\sqrt{\left(\log t+\log|2\delta_1 q|\log\left(1+\frac{\log t}{\log\frac{V}{|2\delta_1 q|}}\right)\right)(\kappa_{w,6}\log t+2\kappa_{w,7})}$$

For $\delta_0 = \delta_1 = 1$, $q = 2\cdot 10^6$, $\phi(q) \sim 4\cdot 10^5$, and $V = 10^{15}/\sqrt{q}$ (a good value to examine, as later arguments will show), this is (to five significant figures)

$$\frac{1}{K\cdot\sqrt{2\cdot 10^6}}(69.63208 + 6.41795\log K)$$
$$+\frac{1}{\sqrt{4\cdot 10^5}}\sqrt{\log K + 7.25433 + 15.20180\log\left(1+\frac{7.25433+\log K}{\log 13.12236}\right)}$$
$$\cdot\sqrt{0.89925\log K + 7.24806}$$

if $w = 1$,

$$\frac{1}{K\cdot\sqrt{2\cdot 10^6}}(136.72211 + 12.8359\log K)$$
$$+\frac{1}{\sqrt{4\cdot 10^5}}\sqrt{\log K + 7.25433 + 15.20180\log\left(1+\frac{7.25433+\log K}{\log 13.12236}\right)}$$
$$\cdot\sqrt{0.60428\log K + 4.56485}$$

if $w = 2$. (Larger choices of w are also possible, but preliminary computations make it doubtful that they give better results.)

The choice

$$K = K_w = \begin{cases} 26 & \text{if } w = 1,\\ 81 & \text{if } w = 2\end{cases}$$

seems best or nearly best. We settle for $K_1 = 25$, $K_2 = 49.8$ to give ourselves some leg-room.

[3] An alternative method would be to approximate $\log(1 + (\log x/UV)/(\log V/2q))$ by $(\log x/UV)/(\log V/2q)$ and then use the non-principal branch of the Lambert function.

We set $UV = x/t = x/K_w\sqrt{q\delta_0}$, at least for q considerably smaller than V. (The exact value of q at which the choice of t will change will be set later.) Then (5.39) plus (5.40) is at most

$$\begin{aligned}\frac{x}{\sqrt{|\delta_0|\phi(q)}}&\sqrt{3.21888 + \log|2\delta_1 q|\left(\frac{1}{2} + \log\left(1 + \frac{\log\frac{x}{UV}}{\log\frac{V}{|2\delta_1 q|}}\right)\right)} \\ &\cdot\sqrt{0.44963\log q\delta_1 + 3.61917} + \frac{x}{\sqrt{q\delta_1}}(0.11919\log q\delta_1 + 1.74931)\end{aligned} \tag{5.41}$$

for $w = 1$, and

$$\begin{aligned}\frac{x}{\sqrt{|\delta_0|\phi(q)}}&\sqrt{3.91202 + \log|2\delta_1 q|\left(\frac{1}{2} + \log\left(1 + \frac{\log\frac{x}{UV}}{\log\frac{V}{|2\delta_1 q|}}\right)\right)} \\ &\cdot\sqrt{0.30214\log q + 2.54516} + \frac{x}{\sqrt{q\delta_1}}(0.11919\log q\delta_1 + 1.87642)\end{aligned} \tag{5.42}$$

for $w = 2$. We will work with $w = 2$ from now on, since it is now clear that it is the better choice: compare the terms under the second root in (5.41) and (5.42).

Now we must decide how to choose U, V and Q, given our choice of UV.

Our tactic at this point will be to choose $Q \sim x/4y$ (say) and use the $S_{I,2}$ estimates for $q \leq Q/V$ to treat all α of the form $\alpha = a/q + O_{\leq}(1/qQ)$, $q \leq y$. The remaining α then get treated with the (coarser) $S_{I,2}$ estimate for $q > Q/V$, with Q reset to a lower value (call it Q'). If α was not treated in the first go (so that it must be dealt with the coarser estimate) then $\alpha = a'/q' + \delta'/x$, where either $q' > y$ or $\delta' q' > x/Q = 4y$. The value of Q' is set to be smaller than Q both because this is helpful (it diminishes error terms that would be large for large q) and because this is now harmless (since we are no longer assuming that $q \leq Q/V$).

5.2.1. *First choice of parameters; $q \leq y$.* The largest items affected by our choices at this point are

$$56.6115\frac{x}{Q/V}\log^+\frac{2UVw^2}{Q}, \qquad 1.7658q\log 2eq \qquad (\text{from } S_{I,2},\ \epsilon = 1/4), \tag{5.43}$$

$$\kappa_{w,8}\sqrt{\frac{q}{\phi(q)}}\left(1 + 1.15\sqrt{\frac{\log 2q}{\log x/2Uq}}\right)\frac{x}{\sqrt{U}} + \kappa_{w,9}\frac{x}{\sqrt{V}} \qquad (\text{from } S_{II}) \tag{5.44}$$

In addition, we have a relatively mild but important dependence on V in the main term (5.40). We must also respect the condition $q \leq Q/V$, the lower bound on U given by (5.16) and the assumptions made at the beginning of section 5 (e.g. $Q \geq x/U$). It is from this point onwards that we assume $x \geq x_0 = 3.1 \cdot 10^{28}$; this will simplify many of our choices. We also set $w = 2$, and so $K_w = 50$.

The following choices then make sense: we will work with $q \leq y$, where

$$\begin{aligned} &y = 2x^{0.3}, \qquad Q = \frac{x}{4y} = \frac{1}{8}x^{0.7}, \qquad x/UV = K_w\sqrt{q\delta_1} \leq 50\sqrt{2y} = 100\sqrt{x^{0.3}}, \\ &U = \frac{x^{0.45}}{4}, \qquad V = \frac{UV}{U} = \frac{x}{(x/UV)\cdot U} = \frac{4x^{0.55}}{K_w\sqrt{q\delta_1}} \geq \frac{x^{0.4}}{25},\end{aligned} \tag{5.45}$$

where, as before,

$$\delta_1 = \begin{cases} 1 & \text{if } |\delta| \leq 4, \\ |\delta|/2 & \text{if } |\delta| > 4. \end{cases}$$

(Note that $\delta/x \leq 1/qQ$ implies $|q\delta| \leq x/Q = 4y$, and thus $q\delta_1 \leq 2y$.)

Since $Q = x/4y$,

$$\begin{aligned} Q/V &= \frac{QU}{UV} = \frac{K_w QU\sqrt{q\delta_1}}{x} = K_w \frac{U}{4y}\sqrt{q\delta_1} = \frac{50}{4}\frac{x^{0.45}}{4y}\sqrt{q\delta_1 y} \\ &= \frac{50}{4}\frac{\sqrt{y}}{4\cdot 2^{3/2}}\sqrt{q\delta_1 y} \geq \frac{50}{32\sqrt{2}}q = 1.1048\ldots q > q. \end{aligned}$$

Moreover, $U/(x/UV) \geq x^{0.3}/400 \geq 8.81\cdot 10^5 > 5\cdot 10^5$, and so (5.16) holds.

It is easy to check that

(5.46)
$$U < x/4.33, \quad 2\cdot 10^6 \leq V < x/4, \quad UV \leq x, \quad Q \geq \sqrt{ex}, \quad Q \geq \max(U, x/U),$$

as stated at the beginning of section (5). We will also find it convenient that

(5.47)
$$\begin{aligned} \frac{V}{2eq} &\geq \frac{V}{2ey} \geq \frac{x_0^{0.4}/25}{4ex_0^{0.3}} = \frac{x_0^{0.1}}{100e} = 2.599\ldots > 1, \\ \frac{V}{e|\delta q|} &\geq \frac{V}{4ey} = \frac{x_0^{0.1}}{200e} = 1.2996\ldots > 1. \end{aligned}$$

The first type I bound is

$$\begin{aligned} |S_{I,1}| &\leq \frac{x}{q}\cdot\min\left(1, \frac{48}{(2\pi\delta)^2}\right)\cdot\left(1.4 + \frac{q}{\phi(q)} + \left(\log\frac{x}{2U}\right)\lambda_1\right) \\ &\quad + 1.7651\cdot y\log(2ey)\log x + 0.88255\cdot x^{0.45}\left(\log(118.4x^{0.55})\log(2ex^{0.55}) + 1\right) \\ &\quad + 3.5683\left(\frac{6}{16x^{0.1}q}\log(4e^{1/2}x^{0.55}) + \frac{4}{e}\right) \\ &\leq \frac{x}{\phi(q)\delta_2^2}\cdot\left(2.4 + \frac{2\log(2x^{0.55})}{\max\left(2, \log\frac{x^{0.45}}{8q^2}\right)}\right) + 0.073x^{0.6}, \end{aligned}$$

where $\delta_2 = \max(1, (\pi/2\sqrt{3})\delta)$ and λ_1 is as in (5.6). It is easy to show that, for q fixed,

(5.48)
$$\frac{2\log(2x^{0.55})}{\max\left(2, \log\frac{x^{0.45}}{8q^2}\right)}$$

attains its maximum at $x = (8e^2q^2)^{1/0.45}$, and this maximum equals

$$\log\left((8e^2q^2)^{0.55/0.45}/2\right) = \frac{22}{9}\log q + \left(\frac{22}{9} + \frac{8}{3}\log 2\right);$$

of course, (5.48) is also at most $\log(2x^{0.55})$ in general. Thus

$$|S_{I,1}| \leq \frac{x}{\phi(q)\delta_2^2}\cdot\min\left(\frac{22}{9}\log q + 6.693, \frac{11}{20}\log x + 3.094\right) + 0.073x^{0.6}.$$

The second type I bound is (by (5.12), with $\epsilon = 0.06$)

$$\begin{aligned}
|S_{I,2}| \leq\ & \frac{x}{q} \cdot \min\left(1, \frac{48}{(2\pi\delta)^2}\right) \lambda_2 \log(Vq) + \frac{x}{\sqrt{q\delta_1}} \cdot (1.68373 + 0.11443 \log(q\delta_1)) \\
& + 6\left(1.8535 \frac{\frac{x^{0.45}}{4}\left(\frac{x^{0.45}}{4} + q\right)}{qx} \left(\frac{4x^{0.55}}{50\sqrt{q\delta_1}}\right)^2 + 7.137 \frac{x^{0.45}/4}{50\sqrt{q\delta_1}} \log(2x^{0.3})\right) \\
& + 1.7658 \cdot 2x^{0.3} \log\left(4ex^{0.3}\right) + 30.812 \frac{32x^{0.85}}{25\sqrt{q\delta_1}} + 11.3223 \frac{848x^{0.85}}{75\sqrt{q\delta_1}} \log \frac{8x/50}{x^{0.7}/8} \\
& \leq \frac{x}{\phi(q)\delta_2^2} \lambda_2 \log(Vq) + 0.0045 \frac{x}{q^2\delta_1} + 0.001x^{0.6} \\
& + \frac{x}{\sqrt{q\delta_1}} \cdot \left(1.63303 + 0.11076 \log(q\delta_1) + \frac{38.406 \log x + 71.042}{x^{0.15}}\right)
\end{aligned}$$

where λ_2 is as in (5.10). We simplify the last line by

$$1.63303 + \frac{38.406 \log x + 71.042}{x^{0.15}} \leq 1.63303 + \frac{38.406 \log x_0 + 71.042}{x_0^{0.15}} < 1.771.$$

Note now that

$$M_V = \max\left(\min\left(U, \frac{Q}{2wV}\right), 1\right) = \frac{Q}{4V} = \frac{K_w U}{16y} \sqrt{q\delta_1} = \frac{50}{16 \cdot 8} \sqrt{q\delta_1} x^{0.15}.$$

Thus

$$\begin{aligned}
\frac{\phi(q)}{q} \lambda_2 \log(Vq) &\leq \frac{2\log(Vq)}{\max\left(2, \log \frac{M_V}{2q^2}\right)} \\
&\leq \frac{2\log\left(\frac{4x^{0.55}\sqrt{q}}{50\sqrt{\delta_1}}\right)}{\max\left(2, \log \frac{50x^{0.15}\sqrt{\delta_1}}{256q^{3/2}}\right)} \leq \frac{2\log\left(\frac{4x^{0.55}\sqrt{q}}{50}\right)}{\max\left(2, \log \frac{25x^{0.15}}{128q^{3/2}}\right)}.
\end{aligned} \tag{5.49}$$

For $x \leq (e^2 \cdot 128q^{3/2}/25)^{1/0.15}$, this is at most

$$\log\left(\frac{4x^{0.55}\sqrt{q}}{50}\right) \leq \log\left(\frac{4(e^2 \cdot 128q^{3/2}/25)^{0.55/0.15}\sqrt{q}}{50}\right) \leq 6\log q + 10.796.$$

For $x \geq (e^2 \cdot 128q^{3/2}/25)^{1/0.15}$, the expression (5.49) is of the form $c(t+a)/(t+b)$ (with $t = \log x$), and hence is monotonic on t (and thus on x); hence it adopts its maximum either at $x = (e^2 \cdot 128q^{3/2}/25)^{1/0.15}$ or at $x = \infty$. Therefore, its maximum is

$$\max\left(\frac{0.55}{0.15}, \log\left(\frac{4\left(\frac{e^2 \cdot 128q^{3/2}}{25}\right)^{0.55/0.15}\sqrt{q}}{50}\right)\right) = 6\log q + 10.796.$$

Hence

$$\begin{aligned}
|S_{I,2}| \leq\ & \frac{x}{\phi(q)\delta_2^2} \min\left(\log\left(\frac{4x^{0.55}\sqrt{q}}{50}\right), 6\log q + 10.796\right) \\
& + \frac{x}{\sqrt{q\delta_1}} \cdot (1.771 + 0.11076 \log(q\delta_1)) + 0.0045 \frac{x}{q^2\delta_1} + 0.001x^{0.6}.
\end{aligned}$$

For type II, we have to consider two cases: (a) $|\delta|$ general, and (b) $|\delta| \geq 4$. We have $q < V/2e$ and $|\delta q| < V/e$ thanks to (5.47). We apply (5.34) with $\theta = e$, and obtain

$$\begin{aligned}|S_{II}| &\leq \frac{x}{\sqrt{\phi(q)}} \cdot \sqrt{3.91202 + \left(\frac{1}{2} + \log\left(1 + \frac{\log 50\sqrt{q}}{\log \frac{V}{2q}}\right)\right)\log 2q} \\ &\cdot \sqrt{0.30214 \log q + 2.6202} \\ &+ 5.4806\sqrt{\frac{q}{\phi(q)}}\left(1 + 1.15\sqrt{\frac{\log 2y}{\log x/2Uy}}\right)\frac{2x}{\sqrt{x^{0.45}}} + 3.9086\frac{x}{\sqrt{\frac{x^{0.4}}{25}}} \\ &\leq \frac{x}{\sqrt{\phi(q)}} \cdot \sqrt{3.91202 + C_{x,q}\log 2q} \cdot \sqrt{0.30214 \log q + 2.6202} \\ &+ 4.8971\sqrt{\frac{q}{\phi(q)}}x^{0.8} + 19.543 x^{0.8},\end{aligned}$$

where we define

$$C_{x,t} := \frac{1}{2} + \log\left(1 + \frac{\log 50\sqrt{t}}{\log \frac{x^{0.55}}{25t^{3/2}}}\right)$$

for t such that $25t^{3/2} \leq x^{0.55}/2e$ (as is always the case here, thanks to $q \leq V/2e$).

In case (b), we use (5.37) (again with $\theta = e$) and obtain

$$\begin{aligned}|S_{II}| &\leq \frac{2x}{\sqrt{|\delta|\phi(q)}} \cdot \sqrt{3.91202 + C_{x,\delta_1 q}\log|\delta q|} \cdot \sqrt{0.30214 \log \delta_1 q + 2.6202} \\ &+ \left(\frac{4\sqrt{\kappa_{w,1}}}{\sqrt{(1-y/Q)}}(\sqrt{\log V} + \sqrt{1/\log V}) + \kappa_{w,9}\right)\frac{x}{\sqrt{V}} \\ &+ \kappa_{w,2}\sqrt{\frac{q}{\phi(q)}}\left(\frac{1}{\sqrt{U}} + \frac{1}{\sqrt{8(1-y/Q)Q}}\log(100x^{0.15})\right) \cdot \sqrt{\frac{\log V}{\log V/4y}} \cdot x \\ &\leq \frac{x}{\sqrt{\delta_0\phi(q)}} \cdot \sqrt{3.91202 + C_{x,\delta_1 q} \cdot \log 2\delta_1 q} \cdot \sqrt{0.30214 \log \delta_1 q + 2.6202} \\ &+ \left(77.339 + \sqrt{\frac{q}{\phi(q)}} \cdot 4.5493\right)x^{0.8}.\end{aligned}$$

Now note the fact ([RS62, Thm. 15]) that $q/\phi(q) < \digamma(q)$, where

$$\digamma(q) = e^{\gamma}\log\log q + \frac{2.50637}{\log\log q}. \tag{5.50}$$

It is easy to show that $\digamma(q) \leq \digamma(y) = \digamma(2x^{0.3})$ and that $x \mapsto x^{0.05}/\sqrt{g(2x^{0.3})}$ is increasing for $x \geq x_0$ (indeed for $x \geq 1$). Hence

$$\max\left(19.543 + 4.8971\sqrt{\frac{q}{\phi(q)}}, 77.339 + 4.5493\sqrt{\frac{q}{\phi(q)}}\right) \leq 3.33575 x^{0.05}.$$

Hence, the total bound for $q \leq y$ is at most

$$\begin{aligned}(5.51)\quad &\frac{x}{\sqrt{\delta_0 \phi(q)}} \cdot \sqrt{3.91202 + C_{x,\delta_1 q} \cdot \log 2\delta_1 q} \cdot \sqrt{0.30214 \log \delta_1 q + 2.6202} \\ &+ \frac{x}{\sqrt{q\delta_1}} \cdot (1.771 + 0.11076 \log(q\delta_1)) \\ &+ \frac{x}{\phi(q)\delta_2^2} \min\left(\frac{11}{10}\log x + \frac{1}{2}\log q + 0.5683, \frac{76}{9}\log q + 17.489\right) \\ &+ 0.0045\frac{x}{\delta_1 q^2} + 3.3358x^{0.85},\end{aligned}$$

where

$$C_{x,t} = \frac{1}{2} + \log\left(1 + \frac{\log 50\sqrt{t}}{\log \frac{x^{0.55}}{25t^{3/2}}}\right)$$

$$\delta_0 = \max(1, |\delta|/4), \qquad \delta_1 = \begin{cases} 1 & \text{if } |\delta| \leq 4 \\ |\delta|/2 & \text{if } |\delta| > 4, \end{cases}$$

$$\delta_2 = \max\left(1, \frac{\pi}{2\sqrt{3}}\delta\right)$$

Note that $t \mapsto C_{x,t}$ is an increasing function of t, and thus attains its maximum on $[0, 2y]$ at $t = 2y$. (We recall that $\delta_1 q \leq 2y$.) Both $x \mapsto C_{x,2y}$ and $x \mapsto C_{x,t}$ (fixed t) are decreasing functions. For $x = x_0 = 3.1 \cdot 10^{28}$, we have $C_{x,2\cdot 10^6} = 1.196\ldots$, $C_{x,y} = 2.463\ldots$ and $C_{x,2y} = 3.021\ldots$.

5.2.2. *Second choice of parameters.* If, with the original choice of parameters, we obtained $q > y = 2x^{0.3}$, we now reset our parameters (Q, U and V). We will do so in such a way as to minimize the terms of size about UV. (This makes sense because q is large, and thus the terms proportional to $\frac{1}{\phi(q)}$ are already fairly small.) It would not be sensible to make UV much smaller than Q, since the error terms proportional to Q would then dominate.

We choose

$$V = 2 \cdot x^{0.3}, \qquad U = 500x^{0.35}, \qquad Q = \frac{x}{U} = \frac{x^{0.65}}{500}.$$

Thus (5.16) holds (as an equality). The conditions in (5.46) also hold.

Write $\alpha = a/q + \delta/x$ for the new approximation; we must have either $q > y$ or $|\delta| > 4y/q$, since otherwise a/q would already be a valid approximation under the first choice of parameters.

Let λ_1 be as in (5.6). Clearly $\lambda_1 \leq 1$. Hence, by (5.8),

$$\begin{aligned}|S_{I,1}| \leq &\frac{x}{q\delta_2^2} \cdot \left(1.4 + \frac{q}{\phi(q)} + \log\frac{x}{2U}\right) \\ &+ 1.7651Q \log 2eQ \log x + 2 \cdot 1.7651U\left(\log\frac{59.1792x}{2U}\log\frac{ex}{2U} + 1\right) \\ &+ 3.5683\left(\frac{6U^2}{xy}\log\frac{e^{1/2}x}{U} + \frac{4}{e}\right) \\ \leq &\frac{x}{q\delta_2^2} \cdot \left(\frac{q}{\phi(q)} + 0.65\log x - 5.5\right) + 2.961x^{2/3}.\end{aligned}$$

By (3.15) with $\epsilon = 0.001$,

$$\begin{aligned}
|S_{I,2}| &\leq \frac{x}{q\delta_2^2}\log(Vq) + 4\log 2\cdot\frac{4UV}{\pi}\left(\log\frac{2UV}{e}\log\frac{8ecx}{UV} - 1\right)\\
&+ 4\log 2\cdot\frac{8.008UV}{\pi}\left(\log 2UV\log\frac{2ecx}{UV} + \log 2ecx\right)\\
&+ 4\log 2\cdot\left(\frac{4Q}{\pi}\log(UV+Q)\log 16cx + \frac{2Q}{\pi}\log 2eQ\log Q\right)\\
&+ (\pi^2 - 4)\frac{48}{(2\pi)^2}\frac{3}{x}\left(1.03883\frac{QV}{8} + \frac{\log V}{2Q}\cdot\left(\frac{Q}{4}\right)^2 + \frac{U^2+U}{2}(\log Q)Q\right)\\
&+ 4\log 2\cdot\left(\frac{12cx}{Q} + \frac{3003cx}{Q}\log\frac{8UV}{Q}\right)\log 2UV\\
&\leq \frac{x}{q\delta_2^2}(0.3\log x + \log q + \log 2) + 20.563x^{0.85}.
\end{aligned}$$

Now we must estimate S_{II}. There are two cases: (a) $|\delta| < 4$ and (b) $|\delta| \geq 4$. As we said before, we must have $q > y$ or $|\delta q| > 4y$, and thus in case (a) we certainly have $q > y$. Since $y = 2x^{0.3} = V$, this implies, obviously, that $q > V > V/2e$. Hence we can treat (a) by applying (5.35) with $\theta = e$, provided that $q \leq x/(2eU)$. Let us bound some of the terms in (5.35). First of all,

(5.52)
$$\begin{aligned}
&2\kappa_{w,2}\sqrt{\frac{q}{\phi(q)}}\left(1 + 1.15\sqrt{\frac{\log 2q}{\log x/2Uq}}\right)\frac{x}{\sqrt{U}}\\
&\qquad < 2\kappa_{w,2}\sqrt{\digamma\left(\frac{x}{2eU}\right)}\left(1 + 1.15\sqrt{\log\frac{x}{eU}}\right)\frac{x}{\sqrt{500x^{0.35}}} < 0.04421x^{0.85},
\end{aligned}$$

where $\digamma(q)$ is as in (5.50). (We are using the fact that second expression in (5.52) divided by $x^{0.85}$ is decreasing for $x > 1.5\cdot 10^{14}$.) In much the same way, we obtain that

$$\kappa_{w,3}\left(\sqrt{2e\log\frac{x}{U}} + \frac{(\log x/U)^{3/2} - (\log V)^{3/2}}{3}\right)\frac{x}{\sqrt{2eU}} < 0.4088x^{0.85} \tag{5.53}$$

using the easily checked fact that the quotient of the left side of (5.53) by the right side of (5.53) is a decreasing function of x for $x > 2.4\cdot 10^{28}$. Since $t \mapsto (\log t)^{3/2}$ is convex-down for $t > 1.7$ and $t \mapsto t/e^{t/2}$ has its maximum at $t = 2$,

(5.54)
$$\begin{aligned}
\left((\log 2eq)^{3/2} - (\log V)^{3/2}\right)\frac{x}{\sqrt{q}} &\leq \frac{3}{2}(\log V)^{1/2}\frac{\log 2eq - \log V}{\sqrt{q}}x\\
&\leq \frac{3}{2}(\log V)^{1/2}\frac{\log e^2V - \log V}{\sqrt{eV/2}}x = 3\sqrt{\frac{2\log V}{eV}}x.
\end{aligned}$$

Hence

(5.55)
$$
\begin{aligned}
|S_{II}| \leq& \frac{x}{\sqrt{\phi(q)}} \cdot \sqrt{\log \frac{x}{2eUq} + \log 2q \log\log \frac{x}{2Uq}} \cdot \sqrt{0.60428 \log \frac{x}{2eUq} + 0.2562}\\
&+ \left(\frac{\kappa_{w,3}}{2\sqrt{e}} \sqrt{\log V} + \kappa_{w,2}\sqrt{\log x/U} + \kappa_{w,9}\right) \frac{x}{\sqrt{V}}\\
&+ 0.04421 x^{0.85} + 0.4088 x^{0.85}\\
\leq& \frac{x}{\sqrt{\phi(q)}} \cdot \sqrt{\log \frac{x}{2Uq} + \log 2q \log\log \frac{x}{2Uq} - 1}\sqrt{0.60428 \log \frac{x}{2Uq} - 0.34808}\\
&+ \left(1.7898\sqrt{\log x} + 3.21681 - \frac{6.7094}{\sqrt{\log x}}\right) x^{0.85},
\end{aligned}
$$

where we are using the fact that $t \mapsto \sqrt{t}$ is convex-down.

Let us now consider the other subcase of (a), namely, $q > x/(2eU)$. Then we use (5.36), and get that

$$
\begin{aligned}
|S_{II}| \leq& (\kappa_{w,2}\sqrt{\log x/U} + \kappa_{w,9}/\sqrt{2}) x^{0.85} + \kappa_{w,3}\sqrt{\log x/U} \frac{x}{\sqrt{U}}\\
&+ \frac{\kappa_{w,3}}{3}((\log x/U)^{3/2} - (\log V)^{3/2}) \left(\sqrt{\frac{exU}{2}} + \frac{x}{\sqrt{U}}\right)\\
\leq& (2.7403\sqrt{\log x/U} + 0.09122\sqrt{\log x} + 2.8653) x^{0.85}\\
\leq& \left(2.3006\sqrt{\log x} + 2.8653 - \frac{4.2366}{\sqrt{\log x}}\right) x^{0.85}.
\end{aligned}
\tag{5.56}
$$

We now study case (b), namely, $|\delta| \geq 4$. We know that $|\delta q| > 4y$; we also know that $|\delta|/x \leq 1/qQ$, and so $q \leq x/(|\delta|Q) \leq x/4Q = U/4$ and $|\delta q| \leq x/Q = U$. We treat (b) by applying (5.38); the second line of (5.38) can be treated just as in (5.54). We obtain

(5.57)
$$
\begin{aligned}
|S_{II}| \leq& \frac{2x}{\sqrt{|\delta|\phi(q)}} \sqrt{\log \frac{x}{eU|\delta q|} + \log|\delta q| \log\log \frac{x}{U|\delta q|}} \cdot \sqrt{\kappa_{w,6} \log \frac{x}{eU|\delta q|} + 2\kappa_{w,7}}\\
&+ \frac{\kappa_{w,2}}{2}\sqrt{\frac{\log V}{eV}} x + \left(\sqrt{\frac{\kappa_{w,1}}{1 - \frac{U}{4Q}}}\left(4\sqrt{V} + \frac{4}{\sqrt{\log V}}\right) + \kappa_{w,9}\right) \frac{x}{\sqrt{V}}\\
&+ \kappa_{w,2}\sqrt{\frac{q}{\phi(q)}} \left(\frac{1}{\sqrt{U}} + \frac{1}{\sqrt{8Q\left(1 - \frac{U}{4Q}\right)}} \log \frac{x}{UV}\right) \sqrt{\log eU} \cdot x\\
\leq& \frac{2x}{\sqrt{|\delta|\phi(q)}} \sqrt{\log \frac{x}{eU|\delta q|} + \log|\delta q| \log\log \frac{x}{U|\delta q|}} \cdot \sqrt{\kappa_{w,6} \log \frac{x}{eU|\delta q|} + 2\kappa_{w,7}}\\
&+ \left(1.95801\sqrt{\log 2x^{0.3}} + 3.1522 + \frac{1.37038}{\sqrt{\log 2x^{0.3}}}\right) x^{0.85}.
\end{aligned}
$$

Now we must add all the contributions. We aim at a bound of the form $C(\log\log x \log x)x^{0.85}$. The terms with $x/q\delta_2^2$ in front give a total of

(5.58)
$$\frac{x}{q\delta_2^2}\left(0.95\log x + \log q + \frac{q}{\phi(q)} + (\log 2 - 5.5)\right) \leq \frac{x}{y}\cdot 1.54024 \leq 0.77012x^{0.7}.$$

Hence

$$\begin{aligned}|S_{I,1}| + |S_{I,2}| &\leq 0.77012x^{0.7} + 2.951x^{2/3} + 20.563x^{0.85}\\ &\leq 0.074922(\log\log x)(\log x)x^{0.85}.\end{aligned}$$

The term in (5.55) with $x/\sqrt{\phi(q)}$ in front can be bounded by

$$x\sqrt{\frac{F(q)}{q}}\sqrt{\log\frac{Q}{2eq} + \log 2q\log\log\frac{Q}{2q}}\sqrt{\kappa_{w,6}\log\frac{Q}{2eq} + 2\kappa_{w,7}};$$

this is decreasing for $q \in [y, Q/2e]$, and thus is bounded by its value at $q = y$, which is at most

$$0.160884(\log\log x)(\log x)x^{0.85}.$$

The other terms in (5.55) add up to at most

$$0.061521(\log\log x)(\log x)x^{0.85}.$$

We can easily bound all of (5.56) by

$$0.076427(\log\log x)(\log x)x^{0.85}.$$

Recalling that $|\delta q| > 4y$ in case (b), we see that the expression in (5.57) with $2x/\sqrt{|\delta|\phi(q)}$ in front is at most

$$0.157099(\log\log x)(\log x)x^{0.85},$$

and the other terms in (5.57) add up to at most

$$0.044793(\log\log x)(\log x)x^{0.85}.$$

Hence (5.55) is worst, and the total bound is at most

(5.59)
$$0.297327(\log\log x)(\log x)x^{0.85},$$

where we use the fact that $0.297327 = 0.074922 + 0.160884 + 0.061521$.

5.3. **Conclusion.**

Proof of main theorem. We have shown that $|S_\eta(\alpha, x)|$ is at most (5.51) for $q \leq 2x^{0.3}$ and at most (5.59) for $q > 2x^{0.3}$. It remains to simplify (5.51) slightly. Let $q_0 = 2\cdot 10^6$.

$$\rho = \frac{3.91202 + C_{x_0,q_0}\cdot\log 2q_0}{0.30214\log q_0 + 2.6202}$$

By the geometric mean/arithmetic mean inequality,

$$\sqrt{3.91202 + C_{x,\delta_1 q}\cdot\log 2\delta_1 q}\cdot\sqrt{0.30214\log\delta_1 q + 2.6202}$$

is at most

$$\begin{aligned}
&\frac{1}{2}\left(\frac{1}{\sqrt{\rho}}(3.91202 + C_{x,\delta_1 q}\cdot \log 2\delta_1 q) + \sqrt{\rho}\cdot (0.30214\log \delta_1 q + 2.6202)\right)\\
&\leq 3.2423 + \frac{1}{2}\left(\frac{1}{\sqrt{\rho}} C_{x,\delta_1 q} + \sqrt{\rho}\cdot 0.30214\right)\log 2\delta_1 q\\
&\leq 3.2423 + \left(0.4091 + 0.28151 \log\left(1 + \frac{\log 50\sqrt{\delta_1 q}}{\log \frac{x^{0.55}}{25(\delta_1 q)^{3/2}}}\right)\right)\log 2\delta_1 q.
\end{aligned}$$

Hence, for $q \leq 2x^{0.3}$,

$$\begin{aligned}
|S_\eta(\alpha,x)| \leq\ & \frac{3.2423 + R_{x,\delta_1 q}\log 2\delta_1 q}{\sqrt{\delta_0 \phi(q)}}\cdot x + \frac{1.6943 + 0.1108\log 2\delta_1 q}{\sqrt{\delta_0 q}}\cdot x\\
&+ \min\left(\frac{11}{10}\log x + \frac{1}{2}\log q + 0.5683, \frac{76}{9}\log q + 17.489\right)\frac{x}{\phi(q)\delta_2^2}\\
&+ 0.0045\frac{x}{\delta_1 q^2} + 3.336 x^{0.85},
\end{aligned}$$

where

$$R_{x,t} = 0.4091 + 0.28151\log\left(1 + \frac{\log 50\sqrt{t}}{\log \frac{x^{0.55}}{25 t^{3/2}}}\right).$$

□

In practice, the simplification of (5.51) leads to a loss of less than 0.05 percent in the worst case (for $x \sim x_0$), and much less than that in the sensitive range near $q = 2\cdot 10^6$.

6. The integral over the minor arcs

We have obtained bounds for $S_\eta(x,\alpha)$, where α is in the union $\mathfrak{m}\subset S^1$ of all minor arcs. We must now use these bounds to estimate the total contribution of the minor arcs

$$\int_{\mathfrak{m}} S_{\eta_0}(x,\alpha)\left(S_{\eta_1}(x,\alpha)\right)^2 e(-N\alpha)\, d\alpha \tag{6.1}$$

to the integral over all of S^1, which gives us the (weighted) number of representations of N as a sum of three primes:

$$\int_{S^1} S_{\eta_0}(x,\alpha)\left(S_{\eta_1}(x,\alpha)\right)^2 e(-N\alpha)\, d\alpha = \sum_{n_1+n_2+n_3=N}\eta_0\left(\frac{n_1}{x}\right)\eta_1\left(\frac{n_2}{x}\right)\eta_1\left(\frac{n_3}{x}\right).$$

Here η_0 is as in (1.4), i.e., it is the smoothing we have been using all along; η_1 is an arbitrary smoothing. (It would make sense to choose η_1 so as to maximise the major-arcs contribution, or so as to make that contribution easy to compute.) One would usually set x equal to N times a constant to be optimised.

The traditional procedure here is to bound (6.1) by an $\ell_\infty\cdot\ell_2$ estimate:

$$\left|\int_{\mathfrak{m}} S_{\eta_0}(x,\alpha)\left(S_{\eta_1}(x,\alpha)\right)^2 e(-N\alpha)\, d\alpha\right| \leq (\max_{\alpha\in\mathfrak{m}} S_{\eta_0}(x,\alpha))\cdot |S_{\eta_1}(x,\alpha)|_2^2.$$

This introduces a factor of $|S_{\eta_1}(\lambda,\alpha)|_2^2 \sim C\log N$, C a constant. We will use a technique due to Heath-Brown,[4] based, in turn, on a lemma of Montgomery's we had the chance to use before. Asymptotically, this replaces $C\log N$ by a constant; it is this fact that ensures that only a bounded number of L-functions will have to be checked for the major-arc treatment. For the sensitive range consisting of values of x not much larger than $x_0 = 3.1\cdot 10^{28}$, the improvement will be more slight.

Lemma 6.1 (Heath-Brown)**.** *Let $S(\alpha)=\sum_n a_n e(\alpha n)$, where $a_n\in\mathbb{C}$ has compact support contained in $(\sqrt{x},\infty)$ and $a_n = 0$ whenever n is divisible by any prime $p\le r$. Let*

$$\mathfrak{m}_{y,r} = \bigcup_{q\le r} \bigcup_{\substack{a \bmod q\\ \gcd(a,q)=1}} \left(\frac{a}{q}-\frac{r}{yq}, \frac{a}{q}+\frac{r}{yq}\right) \tag{6.2}$$

for $r\ge 1$, $y\in(2r^2, 2xr^2]$. Then

$$\int_{\mathfrak{m}_{y,r}} |S(\alpha)|^2\, d\alpha \le \left(\sum_{\substack{q\le \frac{\sqrt{y/2}}{r}\\ p|q\Rightarrow p>r}} \frac{\mu(q)^2}{\phi(q)}\right)^{-1} \cdot \sum_n |a_n|^2. \tag{6.3}$$

There is a minor difference between this and the formulation in [Tao, Lemma 4.6] – namely, here we allow the length of the intervals to vary. Compare to the idea of the weighted large sieve in [MV73], which, like the method here, takes advantage of the unequal spacing of Farey fractions.

Proof. Apply Montgomery's lemma [IK04, Lem. 7.15] to the sequence $a_n e(\alpha n)$ with $\alpha\in\mathbb{R}/\mathbb{Z}$, $\omega(p)=1$. We obtain

$$\frac{1}{\phi(q)}\left|\sum_n a_n e(\alpha n)\right|^2 \le \sum_{\substack{a \bmod q\\ \gcd(a,q)=1}} \left|\sum_n a_n e((\alpha + a/q)n)\right|^2$$

for any product q of distinct primes $p\le\sqrt{x}$. Letting $R=\sqrt{y/2}/r\le\sqrt{x}$ and summing over all square-free $q\le R$ coprime to all primes $\le r$, we obtain

$$\left|\sum_n a_n e(\alpha n)\right|^2 \le \frac{1}{L} \sum_{\substack{q\le R\\ q \text{ sq-free}\\ p|q\Rightarrow p>r}} \sum_{\substack{a \bmod q\\ \gcd(a,q)=1}} \left|\sum_n a_n e((\alpha+a/q)n)\right|^2, \tag{6.4}$$

where $L = \sum_{q\le R: p|q\Rightarrow p>r} \mu(q)^2/\phi(q)$.

Let $q_1, q_2\le r$, $a_1\le q_1$, $a_2\le q_2$ with $\gcd(a_i,q_i)=1$ and $a_1/q_1\ne a_2/q_2$. For any $q_1', q_2'\le R$, $a_1'\le q_1'$, $a_2'\le q_2'$ with $\gcd(a_i', q_i')=1$ and $\gcd(q_i', q_1q_2)=1$, the angles $a_1/q_1 + a_1'/q_1'$ and $a_2/q_2+a_2'/q_2'$ are separated by at least $1/q_1q_2R^2$.

[4]Personal communication (2006). The technique has since been described in [Tao, §4]. Granville and Balog previously suggested a plausible alternative, namely, using mollifiers as in [Bal90].

Since $R = \sqrt{y/2}/r$, we know that $1/q_1 q_2 R^2 = 2r^2/yq_1q_2 \geq (2/y)\max(r/q_1, r/q_2)$. Hence the intervals

$$\left(\frac{a_1}{q_1} + \frac{a_1'}{q_1'} - \frac{r}{yq_1}, \frac{a_1}{q_1} + \frac{a_1'}{q_1'} + \frac{r}{yq_1}\right), \qquad \left(\frac{a_2}{q_2} + \frac{a_2'}{q_2'} - \frac{r}{yq_2}, \frac{a_2}{q_2} + \frac{a_2'}{q_2'} + \frac{r}{yq_2}\right)$$

are disjoint. This implies that all translations $\mathfrak{m}_{y,r} + a/q$ (with $q \leq R$ square-free and coprime to all primes $\leq r$, and a coprime to q) are disjoint.

Integrating and summing (6.4) over all intervals in $\mathfrak{m}_{y,r}$, we obtain

$$\begin{aligned}\int_{\mathfrak{m}_{y,r}} |S(\alpha)|^2\, d\alpha &\leq \frac{1}{L} \sum_{\substack{q\leq R\\ q \text{ sq-free}\\ p|q\Rightarrow p>r}} \sum_{\substack{a \bmod q\\ \gcd(a,q)=1}} \int_{\mathfrak{m}_{y,r}+\frac{a}{q}} |S(\alpha)|^2 d\alpha\\ &\leq \frac{1}{L}\int_{\mathbb{R}/\mathbb{Z}} |S(\alpha)|^2\, d\alpha \leq \frac{1}{L}\sum_n |a_n|^2.\end{aligned}$$

□

Later we will need explicit lower bounds on the sum on q in (6.3). By (4.40) (an estimation standard in sieve theory; see, e.g., [Gre01, pp. 63-64]),

$$\sum_{\substack{q\leq R\\ p|q\Rightarrow p>r}} \frac{\mu(q)^2}{\phi(q)} \geq \prod_{p\leq r}\left(1-\frac{1}{p}\right)\cdot \sum_{d\leq R}\frac{1}{d}. \tag{6.5}$$

Also,

$$\prod_{p\leq r}\left(1-\frac{1}{p}\right)\cdot \sum_{d\leq R}\frac{1}{d} > \frac{\log R + \gamma - 1/R}{e^\gamma\left(\log r + \frac{1}{\log r}\right)} \tag{6.6}$$

for $R \geq 1$, where we are bounding the denominator by [RS62, Thm. 8]. Recall $R = \sqrt{y/2}/r$ in our case, and so $\log R = (\log(y/2))/2 - \log r$.

If $R \leq r$, then we obviously cannot do better than the trivial bound

$$\sum_{\substack{q\leq R\\ p|q\Rightarrow p>r}} \frac{\mu(q)^2}{\phi(q)} \geq 1. \tag{6.7}$$

If $R > r$, we can also use the bound

$$\sum_{\substack{q\leq R\\ p|q\Rightarrow p>r}} \frac{\mu(q)^2}{\phi(q)} \geq 1 + \sum_{r<p\leq R}\frac{1}{p} \geq 1 + \log\frac{\log R}{\log r} - \frac{1}{(\log r)^2}, \tag{6.8}$$

(for $r \geq 286$, by [RS62, Thm. 5]), which is better than (6.5) when $r < R \leq r^4$ or so. Further refinements are possible if $R > r^2$; (6.5) is optimal only asymptotically as $(\log R)/(\log r) \to \infty$.

Lemma 6.2. *Let $f, g : \{a, a+1, \dots, b\} \to \mathbb{R}_0^+$, where $a, b \in \mathbb{Z}^+$. Assume that, for all $x \in [a,b]$,*

$$\sum_{a\leq n\leq x} f(n) \leq F(x), \tag{6.9}$$

where $F : [a, b] \to \mathbb{R}$ is continuous, piecewise differentiable and non-decreasing. Then

$$\sum_{n=a}^{b} f(n) \cdot g(n) \leq (\max_{n \geq a} g(n)) \cdot F(a) + \int_a^b (\max_{n \geq u} g(n)) \cdot F'(u) du.$$

Proof. Let $S(n) = \sum_{m=a}^{n} f(m)$. Then, by partial summation,

$$\sum_{n=a}^{b} f(n) \cdot g(n) \leq S(b)g(b) + \sum_{n=a}^{b-1} S(n)(g(n) - g(n+1)). \tag{6.10}$$

Let $h(x) = \max_{x \leq n \leq b} g(n)$. Then h is non-increasing. Hence (6.9) and (6.10) imply that

$$\begin{aligned}\sum_{n=a}^{b} f(n)g(n) &\leq \sum_{n=a}^{b} f(n)h(n)\\ &\leq S(b)h(b) + \sum_{n=a}^{b-1} S(n)(h(n) - h(n+1))\\ &\leq F(b)h(b) + \sum_{n=a}^{b-1} F(n)(h(n) - h(n+1)).\end{aligned}$$

In general, for $\alpha_n \in \mathbb{C}$, $A(x) = \sum_{a \leq n \leq x} \alpha_n$ and F continuous and piecewise differentiable on $[a, x]$,

$$\sum_{a \leq n \leq x} \alpha_n F(x) = A(x)F(x) - \int_a^x A(u)F'(u)du. \qquad (\textit{Abel summation})$$

Applying this with $\alpha_n = h(n) - h(n+1)$ and $A(x) = \sum_{a \leq n \leq x} \alpha_n = h(a) - h(\lfloor x \rfloor + 1)$, we obtain

$$\begin{aligned}&\sum_{n=a}^{b-1} F(n)(h(n) - h(n+1))\\ &\quad = (h(a) - h(b))F(b-1) - \int_a^{b-1} (h(a) - h(\lfloor u \rfloor + 1))F'(u)du\\ &\quad = h(a)F(a) - h(b)F(b-1) + \int_a^{b-1} h(\lfloor u \rfloor + 1)F'(u)du\\ &\quad = h(a)F(a) - h(b)F(b-1) + \int_a^{b-1} h(u)F'(u)du\\ &\quad = h(a)F(a) - h(b)F(b) + \int_a^b h(u)F'(u)du,\end{aligned}$$

since $h(\lfloor u \rfloor + 1) = h(u)$ for $u \notin \mathbb{Z}$. Hence

$$\sum_{n=a}^{b} f(n)g(n) = h(a)F(a) + \int_a^b h(u)F'(u)du.$$

□

Proposition 6.3. *Let $S(\alpha) = \sum_n a_n e(\alpha n)$, $a_n \in \mathbb{C}$, $\{a_n\}$ in $L^1(\mathbb{Z})$. and $\mathfrak{m}_{y,r}$ be as in Lemma 6.1. Let $T : \mathbb{R}/\mathbb{Z} \to \mathbb{C}$. Let $r_0, r_1 \in \mathbb{Z}^+$, $r_1 \geq r_0 \geq 1$ and*

$y \in (2r_1^2, 2xr_0^2]$. Assume that $a_n = 0$ whenever n is divisible by any prime $p \leq r_1$. Let $g : [r_0, r_1] \to \mathbb{R}^+$ be a non-increasing function such that

$$\max_{\alpha \in (\mathbb{R}/\mathbb{Z}) \setminus \mathfrak{m}_{y,r}} |T(\alpha)| \leq g(r) \tag{6.11}$$

for all $r \in \mathbb{R}$, $r_0 \leq r \leq r_1$. Let $H : [r_0, r_1] \to \mathbb{R}^+$ be a non-decreasing, continuous function, differentiable almost everywhere, such that

$$\frac{1}{\sum |a_n|^2} \int_{\mathfrak{m}_{y,r+1}} |S(\alpha)|^2 d\alpha \leq H(r) \tag{6.12}$$

for all $r \in [r_0, r_1]$, and, moreover, $H(r_1) = 1$.

Then

$$\begin{aligned} &\frac{1}{\sum_n |a_n|^2} \int_{(\mathbb{R}/\mathbb{Z}) \setminus \mathfrak{m}_{y,r_0}} |T(\alpha)||S(\alpha)|^2 d\alpha \\ &\qquad \leq g(r_0) \cdot (H(r_0) - I_0) + \int_{r_0}^{r_1} g(r) H'(r) dr, \end{aligned}$$

where

$$I_0 = \frac{1}{\sum_n |a_n|^2} \int_{\mathfrak{m}_{y,r_0}} |S(\alpha)|^2 d\alpha. \tag{6.13}$$

Proof. For $r_0 \leq r < r_1$, let

$$f(r) = \frac{1}{\sum_n |a_n|^2} \int_{\mathfrak{m}_{y,r+1} \setminus \mathfrak{m}_{y,r}} |S(\alpha)|^2$$

and

$$f(r_1) = \frac{1}{\sum_n |a_n|^2} \int_{(\mathbb{R}/\mathbb{Z}) \setminus \mathfrak{m}_{y,r_1}} |S(\alpha)|^2.$$

Then, by (6.11),

$$\frac{1}{\sum_n |a_n|^2} \int_{(\mathbb{R}/\mathbb{Z}) \setminus \mathfrak{m}_{y,r_0}} |S(\alpha)|^2 |T(\alpha)| d\alpha \leq \sum_{r=r_0}^{r_1} f(r) g(r).$$

By (6.12),

$$\begin{aligned} \sum_{r_0 \leq r \leq x} f(r) &= \frac{1}{\sum_n |a_n|^2} \int_{\mathfrak{m}_{y,x+1} \setminus \mathfrak{m}_{y,r_0}} |S(\alpha)|^2 d\alpha \\ &= \left(\frac{1}{\sum_n |a_n|^2} \int_{\mathfrak{m}_{y,x+1}} |S(\alpha)|^2 d\alpha \right) - I_0 \leq H(x) - I_0 \end{aligned} \tag{6.14}$$

for $x \in [r_0, r_1)$. Moreover,

$$\begin{aligned} \sum_{r_0 \leq r \leq r_1} f(r) &= \frac{1}{\sum_n |a_n|^2} \int_{(\mathbb{R}/\mathbb{Z}) \setminus \mathfrak{m}_{y,r_0}} |S(\alpha)|^2 \\ &= \left(\frac{1}{\sum_n |a_n|^2} \int_{\mathbb{R}/\mathbb{Z}} |S(\alpha)|^2 \right) - I_0 = 1 - I_0 = H(r_1) - I_0. \end{aligned}$$

We let $F(x) = H(x) - I_0$ and apply Lemma 6.2 with $a = r_0$, $b = r_1$. We obtain that

$$\begin{aligned}\sum_{r=r_0}^{r_1} f(r)g(r) &\leq (\max_{r\geq r_0} g(r))F(r_0) + \int_{r_0}^{r_1} (\max_{r\geq u} g(r))F'(u)\, du \\ &\leq g(r_0)(H(r_0) - I_0) + \int_{r_0}^{r_1} g(u)H'(u)\, du.\end{aligned}$$

□

Lemma 6.4. *Let $x > x_0 = 3.1 \cdot 10^{28}$. Let*

$$R_{x,r} = c_0 + c_1 \log\left(1 + \frac{\log 50\sqrt{r}}{\log \frac{x^{0.55}}{25r^{3/2}}}\right), \tag{6.15}$$

where $c_0 = 0.4091$, $c_1 = 0.28151$. Then the function

$$r \mapsto \frac{R_x(r)}{r^\beta}$$

is decreasing on $[1, 2x^{0.3}]$ provided that $\beta \geq 0.175$.

Proof. Write

$$f_1(r) = \frac{\log 50\sqrt{r}}{\log \frac{x^{0.55}}{25r^{3/2}}}, \qquad f_2(r) = \frac{\sqrt{r}}{\log 4r\sqrt{\log\log r}}.$$

We need to show that

$$\frac{(c_0 + c_1 \log(1 + f_1(r)))'}{c_0 + c_1 \log(1 + f_1(r))} = \frac{f_1'(r)}{(1 + f_1(r))(c_0 + c_1 \log(1 + f_1(r)))}$$

is less than

$$\frac{f_2'(r)}{f_2(r)} = \frac{\beta}{r}$$

for $r \in [1, 2x^{0.3}]$. We can write $f_1(r) = (\log a)/(3\log b/a)$, where $a = 2500r$ and $b = (5000x^{0.55})^{2/3}$. Therefore,

$$\frac{df_1}{dr}(r) = 2500\frac{d}{da}\frac{\log a}{3(\log b - \log a)} = 2500\frac{\log b}{3(\log b - \log a)^2}\frac{1}{a} = \frac{1}{r}\frac{\log b}{3(\log b/a)^2}$$

and so

$$\begin{aligned}\frac{c_1 f_1'(r)}{(1 + f_1(r))(c_0 + c_1 \log(1 + f_1(r)))} &= \frac{1}{3r}\frac{(\log b)/(\log b/a)^2}{\left(1 + \frac{\log a}{3\log\frac{b}{a}}\right)\left(\frac{c_0}{c_1} + \log\left(1 + \frac{\log a}{3\log\frac{b}{a}}\right)\right)} \\ &= \frac{1}{2r}\frac{\log b}{\log\frac{b}{a}\left(\log\frac{b}{a} + \frac{1}{2}\log b\right)\left(\frac{c_0}{c_1} + \log\left(1 + \frac{\log a}{3\log b/a}\right)\right)} = \frac{1}{2r}\frac{1}{w(\alpha)(\log b)},\end{aligned}$$

where $\alpha = (\log a)/(\log b)$ and

$$w(\alpha) = (1 - \alpha)((1 - \alpha) + 1/2)\left(c_0/c_1 + \log\left(1 + \frac{\alpha}{3(1 - \alpha)}\right)\right).$$

The assumption $x \geq x_0$ implies that $(\log a)/(\log b) < 0.949$. It is easy to check that $\alpha \to w(\alpha)$ decreases on $[0, 1)$ and that $w(0.949) > 0.0963$. Thus,

$$w(\alpha)\log b \geq 0.0963 \cdot 29.732 > 2.863.$$

Hence

$$\frac{1}{2r}\frac{1}{w(\alpha)(\log b)} < \frac{1}{5.726r} \leq \frac{\beta}{r}$$

for $\beta \geq 0.175 > 1/5.726$. □

Lemma 6.5. *Let $x \geq x_0 = 3.1 \cdot 10^{28}$. Then the function $g : (e, 2x^{0.3}] \to \mathbb{R}^+$ given by the following is non-increasing for $r \geq 43$:*

(6.16)
$$g(r) = \left((3.2423 + R_{x,r} \log 2r)\sqrt{F(r)} + 0.1108 \log 2r + 1.6943\right) \frac{x}{\sqrt{r}}$$
$$+ \min\left(\frac{11}{10} \log x + \frac{1}{2} \log r + 0.5728, \frac{76}{9} \log r + 17.4945\right) \frac{F(r)}{r} x + 3.336 x^{0.85},$$

where $R_{x,r}$ is as in (6.15) and

$$F(r) = e^\gamma \log\log r + \frac{2.50637}{\log\log r}. \tag{6.17}$$

Proof. Taking derivatives, we can easily see that
$$r \to \min\left(\frac{11}{10} \log x + \frac{1}{2} \log r + 0.5728, \frac{76}{9} \log r + 17.4945\right) \frac{\log\log r}{r}$$
is a decreasing function for $r \geq 10$,
$$r \to (0.1108 \log 2r + 1.6943) \frac{1}{\sqrt{r}}$$
is a decreasing function for $r \geq 1$,
$$r \to 3.2423 \frac{\sqrt{\log\log r}}{\sqrt{r}}$$
is a decreasing function for $r \geq 14$. These statements remain true if $\log\log r$ is replaced by $F(r)$, since $F(r)/\log\log r$ is a decreasing function for $r \geq 1$.

We can also check easily that
$$r \to \log 2r \frac{\sqrt{\log\log r}}{r^{0.325}}$$
decreases for $r \geq 43$. Together with Lemma 6.4, this implies that
$$r \to R_{x,r} \log 2r \frac{\sqrt{\log\log r}}{\sqrt{r}} = \frac{R_{x,r}}{r^{0.175}} \cdot \log 2r \frac{\sqrt{\log\log r}}{r^{0.325}}$$
is a decreasing function for $r \geq 43$, and so we are done. □

Lemma 6.6. *Let $x \geq x_0 = 3.1 \cdot 10^{28}$. Let*

(6.18)
$$g_2(r) = \left((3.2423 + R_{x,2r} \log 4r)\sqrt{F(r)} + 0.1108 \log 4r + 1.6943\right) \frac{x}{\sqrt{r}}$$
$$+ \min\left(\frac{11}{10} \log x + \frac{1}{2} \log r + 0.5728, \frac{76}{9} \log r + 17.4945\right) \frac{F(r)}{r} x + 3.336 x^{0.85},$$

where $R_{x,r}$ is as in (6.15) and $F(r)$ is as in (6.17). Then, for $r \geq 35000$,
$$g_2(2.25r) \leq g(r),$$
where g is as in (6.16).

Proof. By Lemma 6.4, $R_{x,4.5r} \leq 4.5^{0.175} R_{x,r}$. Since $r \geq 35000$ and $F(r)/\log\log r$ is a decreasing function,
$$\frac{\log 9r \sqrt{F(2.25r)}}{\log 2r \sqrt{F(r)}} < \frac{\log 9r \sqrt{\log\log 2.25r}}{\log 2r \sqrt{\log\log r}} < 1.1528 < \frac{\sqrt{2.25}}{4.5^{0.175}}.$$

This implies that

$$(R_{x,2r_1}\log 4r_1)\frac{\sqrt{F(r_1)}}{\sqrt{r_1}} < (R_{x,r}\log 2r)\frac{\sqrt{F(r)}}{\sqrt{r}}$$

for $r_1 = 2.25r$; the rest is simpler. □

Lemma 6.7. *For any $q \geq 1$ and any $r \geq \max(3,q)$,*

$$\frac{q}{\phi(q)} < F(r)$$

for F as in (6.17).

Proof. Since $F(r)$ is increasing for $r \geq 27$, the statement follows immediately for $q \geq 27$ by [RS62, Thm. 15]:

$$\frac{q}{\phi(q)} < F(q) \leq F(r).$$

For $r < 27$, it is clear that $q/\phi(q) \leq 2\cdot 3/(1\cdot 2) = 3$; it is also to see that $F(r) > 3$ for all $r > e$. □

Theorem 6.8 (Total of minor arcs)**.** *Let $x \geq x_0$, $x_0 = 3.1\cdot 10^{28}$. Let $S_\eta(\alpha, x)$ be as in (1.1); let η_0 be as in (1.4), and let $\eta_1 : [0,\infty) \to [0,\infty)$ be a bounded, piecewise continuous function of fast enough decay (namely, $\lim_{t\to\infty}(\sqrt{t}\log t)\cdot\eta_1^2(t) = 0$ and $|\log(t)\eta_1(t/x)|_1 < \infty$). Assume that the series in (1.1) converges absolutely for $\eta = \eta_1$. Let $\mathfrak{m}_{y,r}$ be as in Lemma 6.1. Let $r_0 \geq 16000$.*

If $r_0 \leq (x/18)^{1/4}$, then

$$\begin{aligned}&\sqrt{\int_{(\mathbb{R}/\mathbb{Z})\setminus\mathfrak{m}_{y,r_0}} |S_{\eta_0}(\alpha,x)||S_{\eta_1}(\alpha,x)|^2 d\alpha}\\ &\leq \sqrt{\left(g(r_0)\cdot(H(r_0+1)-I_0) + \int_{r_0}^{r_1} g(r)H'(r+1)dr\right)\cdot\sum_n \Lambda(n)^2\eta_1(n/x)^2}\\ &+ \max(g(r_0), 0.2974(\log\log x)(\log x)x^{0.85})^{1/2}\\ &\cdot\sqrt{0.7131\int_1^\infty \frac{\log e^2 t}{\sqrt{t}}\cdot\overline{\eta_1}^2\left(\frac{t}{x}\right)dt + 1.03883|\eta_1|_\infty^2\cdot r_1\log r_1}.\end{aligned}\tag{6.19}$$

where $g(r)$ is as in (6.16), $r_1 = (x/18)^{1/4}$, I_0 is as in (6.13),

$$H(r) = \min\left(\frac{e^\gamma\left(\log(r)+\frac{1}{\log(r)}\right)}{\log\varkappa(r)+\gamma-\frac{1}{\varkappa(r)}}, \frac{1}{1+\log\frac{\log\varkappa(r)}{\log r} - \frac{1}{(\log r)^2}}, 1\right)\tag{6.20}$$

and $\varkappa(r) = (x/18)^{1/2}/r$.

For r_0 arbitrary,

$$\begin{aligned}&\frac{1}{|\Lambda(n)\eta_1(n/x)|_2^2}\int_{(\mathbb{R}/\mathbb{Z})\setminus\mathfrak{m}_{y,r_0}} |S_{\eta_0}(\alpha,x)||S_{\eta_1}(\alpha,x)|^2 d\alpha\\ &\leq (1-I_0)\cdot\begin{cases}\max\left(g(r_0), 0.2974(\log\log x)(\log x)x^{0.85}\right) & \text{if } r_0 \leq 2x^{0.3},\\ 0.2974(\log\log x)(\log x)x^{0.85} & \text{if } r_0 > 2x^{0.3}.\end{cases}\end{aligned}\tag{6.21}$$

Proof. Let $Q = x^{0.7}/8$, as in the Main Theorem, and $y = x/9$. Let $\alpha \in (\mathbb{R}/\mathbb{Z}) \setminus \mathfrak{m}_{y,r}$, where $r \geq r_0$. There exists an approximation $\alpha = a/q + \delta/x$ with $q \leq Q$, $|\delta|/x \leq 1/qQ$.

There are three cases:

(1) $q \leq r$. Since α is not in $\mathfrak{m}_{y,r}$, then, by definition (6.2), $|\delta|/x \geq r/yq = 9r/xq$; that is to say, $|\delta| \geq 9r/q$, and, in particular, $|\delta| \geq 9 > 4$. Thus, for δ_0, δ_1 and δ_2 as in (1.3),

$$\delta_0 q = \frac{|\delta|}{4} q, \qquad \delta_1 q = \frac{|\delta|}{2} q, \qquad \delta_2 \phi(q) = \frac{\pi}{2\sqrt{3}} |\delta| q.$$

Recall that $q/\phi(q) < \digamma(\delta_0 q)$ by Lemma 6.7; we note also that $\delta_0 q \leq (x/qQ)/4 \cdot q = x/4Q = 2x^{0.3}$. Therefore, the main theorem implies that $|S_{\eta_0}(x,\alpha)| \leq g_2(\delta_0 q)$, where g_2 is as in (6.18). Now $\delta_0 q = |\delta| q/4 \geq (9r/q)q/4 = 2.25r \geq 2.25 r_0$. By Lemma 6.6 (applicable since $2.25 r_0 \geq 2.25 \cdot 16000 > 35000$) and Lemma 6.5,

$$g_2(\delta_0 q) \leq g\left(\frac{\delta_0 q}{2.25}\right) \leq g(r).$$

Therefore,

$$|S_{\eta_0}(x,\alpha)| \leq g(r). \tag{6.22}$$

(2) $r < q \leq 2x^{0.3}$. Then, by the main theorem and Lemma 6.5,

$$|S_{\eta_0}(x,\alpha)| \leq g(q) \leq g(r). \tag{6.23}$$

(3) $q > 2x^{0.3}$. By the main theorem,

$$|S_{\eta_0}(x,\alpha)| \leq 0.2974 (\log\log x)(\log x) x^{0.85}. \tag{6.24}$$

We set $r = r_0$ and bound $|S_{\eta_0}(x,\alpha)|$ for $\alpha \in \mathfrak{m}_{y,r}$ as above. In consequence,

$$\begin{aligned}
&\frac{1}{\sum_n \Lambda(n)^2 \eta_1(n/x)^2} \int_{(\mathbb{R}/\mathbb{Z})\setminus\mathfrak{m}_{y,r_0}} |S_{\eta_0}(\alpha,x)||S_{\eta_1}(\alpha,x)|^2 d\alpha \\
&\leq \frac{1}{\sum_n \Lambda(n)^2 \eta_1(n/x)^2} \left(\int_{(\mathbb{R}/\mathbb{Z})\setminus\mathfrak{m}_{y,r_0}} |S_{\eta_1}(\alpha,x)|^2 d\alpha\right) \cdot \max_{\alpha\in(\mathbb{R}/\mathbb{Z})\setminus\mathfrak{m}_{y,r_0}} |S_{\eta_0}(\alpha,x)| \\
&\leq (1 - I_0) \cdot \max_{\alpha\in(\mathbb{R}/\mathbb{Z})\setminus\mathfrak{m}_{y,r_0}} |S_{\eta_0}(\alpha,x)| \\
&\leq (1 - I_0) \cdot \begin{cases} \max\left(g(r_0), 0.2974(\log\log x)(\log x)x^{0.85}\right) & \text{if } r_0 \leq 2x^{0.3}, \\ 0.2974(\log\log x)(\log x)x^{0.85} & \text{if } r_0 > 2x^{0.3}. \end{cases}
\end{aligned}$$

Assume now that $r_0 \leq (x/8)^{1/4}$. Let $r_1 = (x/8)^{1/4}$. Write

$$\begin{aligned}
S_{1,\eta_1}(\alpha,x) &= \sum_{p > r_1} (\log p) e(\alpha p) \eta_1(p/x), \\
S_{2,\eta_1}(\alpha,x) &= \sum_{n \text{ non-prime}} \Lambda(n) e(\alpha n) \eta_1(n/x) + \sum_{n \leq r_1} \Lambda(n) e(\alpha n) \eta_1(n/x).
\end{aligned}$$

By the triangle inequality (with weights $|S_{\eta_0}(\alpha,x)|$),

$$\begin{aligned}&\sqrt{\int_{(\mathbb{R}/\mathbb{Z})\setminus\mathfrak{m}_{y,r_0}} |S_{\eta_0}(\alpha,x)||S_{\eta_1}(\alpha,x)|^2 d\alpha}\\ &\quad\leq \sum_{j=1}^{2} \sqrt{\int_{(\mathbb{R}/\mathbb{Z})\setminus\mathfrak{m}_{y,r_0}} |S_{\eta_0}(\alpha,x)||S_{j,\eta_1}(\alpha,x)|^2 d\alpha}.\end{aligned}$$

Clearly,

$$\begin{aligned}&\sqrt{\int_{(\mathbb{R}/\mathbb{Z})\setminus\mathfrak{m}_{y,r_0}} |S_{\eta_0}(\alpha,x)||S_{2,\eta_1}(\alpha,x)|^2 d\alpha}\\ &\leq \max_{\alpha\in(\mathbb{R}/\mathbb{Z})\setminus\mathfrak{m}_{y,r_0}} |S_{\eta_0}(\alpha,x)|^{1/2}\cdot\sqrt{\int_{\mathbb{R}/\mathbb{Z}} |S_{2,\eta_1}(\alpha,x)|^2 d\alpha}\\ &\leq \max_{\alpha\in(\mathbb{R}/\mathbb{Z})\setminus\mathfrak{m}_{y,r_0}} |S_{\eta_0}(\alpha,x)|^{1/2}\cdot\sqrt{\sum_{n \text{ non-prime}} \Lambda(n)^2\eta_1(n/x)^2 + \sum_{n\leq r_1}\Lambda(n)^2\eta_1(n/x)^2}.\end{aligned}$$

Let $\overline{\eta_1}(z) = \sup_{w\geq z}\eta_1(w)$. By [RS62, Thm. 12], [RS62, Thm. 13] and partial summation,

$$\begin{aligned}\sum_{n \text{ non-prime}} \Lambda(n)^2\eta_1(n/x)^2 &\leq \sum_{n \text{ non-prime}} \Lambda(n)^2\overline{\eta_1}(n/x)^2\\ &\leq \int_1^\infty \left(\sum_{\substack{n\leq t\\ n \text{ non-prime}}} \Lambda(n)^2\right)\left(\overline{\eta_1}^2(t/x)\right)' dt\\ &\leq \int_1^\infty (\log t)\cdot 1.4262\sqrt{t}\left(\overline{\eta_1}^2(t/x)\right)' dt\\ &\leq 0.7131\int_1^\infty \frac{\log e^2 t}{\sqrt{t}}\cdot\overline{\eta_1}^2\left(\frac{t}{x}\right)dt,\end{aligned}$$

$$\begin{aligned}\sum_{n\leq r_1}\Lambda(n)^2\eta_1(n/x)^2 &\leq |\eta_1|_\infty^2(\log r_1)\sum_{n\leq r_1}\Lambda(n)\\ &\leq 1.03883|\eta_1|_\infty^2\cdot r_1\log r_1.\end{aligned}$$

It remains to bound

$$\int_{(\mathbb{R}/\mathbb{Z})\setminus\mathfrak{m}_{y,r_0}} |S_{\eta_0}(\alpha,x)||S_{1,\eta_1}(\alpha,x)|^2 d\alpha.$$

We must check that the conditions of Prop. 6.3 hold for $S(\alpha) = S_{1,\eta_1}(\alpha,x)$, $T(\alpha) = S_{\eta_0}(\alpha,x)$. It is clear that $y = x/9 \in (2r_1^2, 2xr_0^2]$. We defined S_{1,η_1} in such a way as to let it be supported on integers coprime to all primes $p\leq r_1$. By Lemma 6.5, $g(r)$ is non-increasing on $[r_0,r_1]$. In particular, $g(r)\geq g(r_1)$. It is easy to show that

$$g(r_1) = g\left((x/8)^{1/4}\right) > 0.2974(\log\log x)(\log x)x^{0.85}$$

for all $x > x_0$. Hence (6.22), (6.23) and (6.24) imply that

$$|S_{\eta_0}(x,\alpha)|\leq g(r)$$

for all $\alpha\in(\mathbb{R}/\mathbb{Z})\setminus\mathfrak{m}_{y,r}$.

Inequality (6.12) (with $H(r+1)$ instead of $H(r)$) holds by Lemma 6.1 and (6.5)–(6.8) (with $R = \sqrt{y/2}/r$). It is clear that $H : [r_0+1, r_1+1] \to \mathbb{R}^+$ is non-decreasing, continuous and differentiable almost everywhere, since it is the minimum of three functions satisfying these three conditions and crossing at finitely many places. It remains to check that $H(r_1+1) = 1$. This follows easily from (6.20) and the fact that $\varkappa(r_1+1) < r_1+1$.

We apply Prop. 6.3, and obtain

$$\begin{aligned}&\int_{(\mathbb{R}/\mathbb{Z})\setminus \mathfrak{m}_{y,r_0}} |S_{\eta_0}(\alpha,x)|\cdot|S_{1,\eta_1}(\alpha,x)|^2\, d\alpha\\ &\leq \left(g(r_0)\cdot(H(r_0+1)-I_0) + \int_{r_0}^{r_1} g(r)H'(r+1)dr\right)\cdot \sum_n \Lambda(n)^2\eta_1(n/x)^2.\end{aligned}$$

□

Let us discuss briefly how to read the statement of Theorem 6.8. The last two lines of (6.19) are negligible. If $r_0 < r_1$, the leading term in $H'(r)$ on the range $[r_0, r_1]$ (save for the rightmost part, where $H'(r) = 0$) is proportional to $1/r \log r$. For $g(r)$ as in (6.16), this implies that $\int_{r_0}^{r_1} g(r)H'(r+1)dr$ is bounded by (essentially) a constant times $g(r_0)/\log r_0$. In contrast, the main term $g(r_0)\cdot H(r_0+1)$ is (in practice) a constant times $g(r_0)$.

The size of the negative, and hence helpful, term $-g(r_0)I_0$ depends on the quality of major-arc estimates, but it is always within $-Cg(r_0)(\log r_0)/\log x$ and 0 for a constant C.

For $x = x_0 = 3.1\cdot 10^{28}$, (6.20) gives us that $H(2\cdot 10^6) = 0.85578\dots$ and $H(4\cdot 10^6) = 0.92115\dots$. Numerical integration gives

$$g(r_0)\cdot H(r_0+1) + \int_{r_0}^{r_1} g(r)H'(r+1)dr = \begin{cases} 0.02316\dots & \text{for } r_0 = 2\cdot 10^6,\\ 0.01773\dots & \text{for } r_1 = 4\cdot 10^6,\end{cases}$$

which is only slightly better than the bounds given more naively by (6.21), namely, $g(r_0) = 0.02379\dots$ (for $r_0 = 2\cdot 10^6$) and $g(r_0) = 0.01782\dots$ (for $r_0 = 4\cdot 10^6$). The eventual major-arc help $-g(r_0)I$ would be the same in each case.

The true power of the improvement of (6.20) over (6.19) becomes clear only for x much larger than x_0. As x increases and r remains fixed, $H(r)$ asymptotes to $2e^\gamma(\log r)/(\log x)$. The denominator cancels the factor of $\log x$ implicit in $\sum_n \Lambda(n)^2\eta_1(n/x)^2$ (see (6.19)). It is because of this that it will be enough to check major arcs contributions up to a constant r_0, even as $x\to\infty$. See Table 2, which gives values of

$$\left(g(r_0)\cdot H(r_0+1) + \int_{r_0}^{r_1} g(r)H'(r+1)dr\right)\cdot \log x \tag{6.25}$$

for variable x and $r_0 = 4\cdot 10^6$. (Prolonging the table after 10^{80} is unnecessary, since $H(r_0)$ is already equal to the first term within min in (6.20), and so $g(r_0)\cdot H(r_0+1)\cdot\log x$ cannot grow.)

There is a middle range (roughly: x between r^6 and r^{10}) for which Theorem 6.8 can be refined further by the inclusion of a fourth term in (6.20). The reason why the middle term in (6.20) is sometimes better than the first term is that $1 + \sum_{r<p\leq z} 1/p$ is a better approximation to $\sum_{n\leq z: p|n\Rightarrow p>r} 1/n$ than $(\log z)/(e^\gamma \log r)$ when z is between r and r^4 (or a little further). When z is between r^2 and r^4, we can also gain something by considering terms of the form $1/p_1p_2$, $r < p_1, p_2 < z/r$, $p_1p_2\leq z$. This can be used to improve slightly on the middle range in Table 2.

x	Bound on minor-arc integral
$3.1 \cdot 10^{28}$	$1.163699\ldots$
10^{29}	$1.162418\ldots$
10^{30}	$1.153979\ldots$
10^{32}	$1.124607\ldots$
10^{34}	$1.090788\ldots$
10^{36}	$1.059498\ldots$
10^{40}	$1.012654\ldots$
10^{50}	$0.975229\ldots$
10^{60}	$0.997130\ldots$
10^{70}	$1.042510\ldots$
10^{80}	$1.028066\ldots$

Table 2. Size of the bound (6.25) for $r_0 = 4 \cdot 10^6$

Appendix A. Computational work on $n = p_1 + p_2 + p_3$ for n small

In order to obtain an n_0 such that every odd number $n \leq n_0$ is the sum of three primes, we need only combined known estimates on the binary Goldbach problem ($n = p_1 + p_2$) and on prime numbers in short intervals. Estimates on primes in short intervals are generally based on numerical verifications of the Riemann hypothesis up to a certain height, i.e., checking that all (non-trivial) zeroes z of the Riemann zeta function up to a height H (meaning: $\Im(z) \leq H$) lie on the critical line $\Re(z) = 1/2$.

The height up to which the Riemann hypothesis has actually been fully verified is not a matter on which there is unanimity. The strongest claim in the literature is in [GD04], which states that the first 10^{13} zeroes of the Riemann zeta function lie on the critical line $\Re(z) = 1/2$. This corresponds to checking the Riemann hypothesis up to height $H = 2.44599 \cdot 10^{12}$. It is unclear whether this computation was or could be easily made rigorous; as pointed out in [SD10, p. 2398], it has not been replicated yet.

Before [GD04], the strongest results were those of the ZetaGrid distributed computing project led by S. Wedeniwski [Wed05]; the method followed in it was more traditional, and should allow rigorous verification involving interval arithmetic. Unfortunately, the results were never formally published. The statement that the ZetaGrid project verified the first $9 \cdot 10^{11}$ zeroes (corresponding to $H = 2.419 \cdot 10^{11}$) is often quoted (e.g., [Bom10, p. 29]); this is the point to which the project had got by the time of Gourdon and Demichel's announcement. Wedeniwski asserts in private communication that the project verified the first 10^{12} zeroes, and that the computation was double-checked (by the same method).

The strongest claim prior to ZetaGrid was that of van de Lune ($H = 3.293 \cdot 10^9$, first 10^{10} zeroes; unpublished). Recently, Platt [Pla] checked the first $1.1 \cdot 10^{11}$ zeroes ($H = 3.061 \cdot 10^{10}$) rigorously, following a method essentially based on that in [Boo06]. Note that [Pla] uses interval arithmetic, which is highly desirable for floating-point computations.

As for the verification of the binary Goldbach conjecture, we will use the work of Oliveira e Silva [OeS12], which (as of April, 2012) states that every even integer $4 \leq n \leq 4 \cdot 10^{18}$ is the sum of two primes. Presumably, this verification involves only integer arithmetic, and thus there are many fewer potential concerns than in the above.

Proposition A.1. *Every odd integer $5 \leq n \leq n_0$ is the sum of three primes, where*

$$n_0 = \begin{cases} 5.90698 \cdot 10^{29} & \textit{if [GD04] is used – } H = 2.44 \cdot 10^{12}, \\ 6.15697 \cdot 10^{28} & \textit{if ZetaGrid results are used } (H = 2.419 \cdot 10^{11}), \\ 1.23163 \cdot 10^{27} & \textit{if [Pla] is used } (H = 3.061 \cdot 10^{10}). \end{cases}$$

Proof. For $n \leq 4 \cdot 10^{18} + 3$, this is immediate from [OeS12]. Let $10^{18} + 3 < n \leq n_0$. We need to show that there is a prime p in $[n - 4 - (n-4)/\Delta, n-4]$, where Δ is large enough for $(n-4)/\Delta \leq 4 \cdot 10^{18} - 4$ to hold. We will then have that $4 \leq n - p \leq 4 + (n-4)/\Delta \leq 4 \cdot 10^{18}$. Since $n - p$ is even, [OeS12] will then imply that $n - p$ is the sum of two primes p', p'', and so

$$n = p + p' + p''.$$

Since $n - 4 > 10^{11}$, the interval $[n - 4 - (n-4)/\Delta, n-4]$ with $\Delta = 28314000$ must contain a prime [RS03]. This gives the solution for $(n-4) \leq 1.1325 \cdot 10^{26}$, since then $(n-4) \leq 4 \cdot 10^{18} - 4$. Note $1.1325 \cdot 10^{26} > e^{59}$.

From here onwards, we use the tables in [Ramd] to find acceptable values of Δ. Since $n - 4 \geq e^{59}$, we can choose

$$\Delta = \begin{cases} 52211882224 & \text{if [GD04] is used (case (a)),} \\ 13861486834 & \text{if ZetaGrid is used (case (b)),} \\ 307779681 & \text{if [Pla] is used (case (c)).} \end{cases}$$

This gives us $(n-4)/\Delta \leq 4 \cdot 10^{18} - 4$ for $n - 4 < e^{r_0}$, where $r_0 = 67$ in case (a), $r_0 = 66$ in case (b) and $r_0 = 62$ in case (c).

If $n - 4 \geq e^{r_0}$, we can choose (again by [Ramd])

$$\Delta = \begin{cases} 146869130682 & \text{in case (a),} \\ 15392435100 & \text{in case (b),} \\ 307908668 & \text{in case (c).} \end{cases}$$

This is enough for $n - 4 < e^{68}$ in case (a), and without further conditions for (b) or (c).

Finally, if $n - 4 \geq e^{68}$ and we are in case (a), [Ramd] assures us that the choice

$$\Delta = 147674531294$$

is valid; we verify as well that $(n_0 - 4)/\Delta \leq 4 \cdot 10^{18} - 4$. $\square$

In the main theorem, I have chosen to include the assumption $x \geq 3.1 \cdot 10^{28}$. This means that, for it to be applied to Goldbach's three-prime problem (as in §6), we must assume n_0 is at least about $1.8 \cdot 3.1 \cdot 10^{28} = 5.58 \cdot 10^{28}$. In other words, we are using the ZetaGrid results (or something slightly weaker), but we never need [GD04].

Downgrading the assumptions so that only [Pla] is used is a fairly easy matter: we only need to modify some parameters starting in 5.2.1. The cost would be a moderate degradation in the bounds (at least about 5 percent) and a slight lengthening of the proof.

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